HUBBARD-STRATONOVICH TRANSFORMATION

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 24.2.

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We start with the ϕ^4 Lagrangian:

$$\mathcal{L}=\frac{1}{2}\left(\partial_\mu\phi\right)^2-\frac{m^2}{2}\phi^2-\frac{g}{8}\phi^4 \tag{1}$$

We now transform this Lagrangian by introducing an extra field σ:

This is known as a Hubbard-Stratonovich transformation, and serves to eliminate the ϕ^4 term.

$$\mathcal{L}'=\mathcal{L}+\frac{1}{2g}\left(\sigma-\frac{g}{2}\phi^2\right)^2 \tag{2}$$

$$=\frac{1}{2}\left(\partial_\mu\phi\right)^2-\frac{m^2}{2}\phi^2+\frac{1}{2}\frac{\sigma^2}{g}-\frac{1}{2}\sigma\phi^2 \tag{3}$$

Note that the ϕ^4 term has been eliminated.

The functional integral to be done now requires two functional integrals: one over the original field ϕ and one over the new field σ. That is, we have (taking the source term $J=0$ in L&B's equation 24.8):

$$Z[0]=\int\mathcal{D}[\phi]\int\mathcal{D}[\sigma]\,e^{i\int d^4x\mathcal{L}'} \tag{4}$$

We can do the integral over σ by considering only those terms in the exponent that contain σ, so we have

$$\int\mathcal{D}[\sigma]\exp\left[\int\left(\frac{i}{2}\frac{\sigma^2}{g}-\frac{i}{2}\sigma\phi^2\right)\right] \tag{5}$$

This integral is in the form of L&B's equation 24.5 if we write it as

$$Z_\sigma\equiv\int\mathcal{D}[\sigma]\exp\left[\frac{i}{2}\int d^4x\,d^4y\,\sigma(x)\frac{1}{g}\sigma(y)-\frac{i}{2}\int d^4x\,\sigma(x)\phi^2(x)\right] \tag{6}$$

We can identify

$$A(x,y)=\frac{1}{g} \tag{7}$$

and

$$b(x) = -\frac{1}{2}\phi^2(x) \tag{8}$$

in equation 24.5. The formal solution of this is

$$Z_\sigma = \frac{B}{\sqrt{\det A}} \exp\left[-\frac{i}{2}\int d^4x\, d^4y\, b(x) A^{-1}(x,y) b(y)\right] \tag{9}$$

where the inverse A^{-1} satisfies (see L&B's equation 24.15 for a similar example)

$$A(x,y) A^{-1}(x,y) = \delta^{(4)}(x-y) \tag{10}$$

In our case, $A(x,y) = \frac{1}{g}$ is just a constant, so we must have

$$A^{-1}(x,y) = g\delta^{(4)}(x-y) \tag{11}$$

Plugging this and 8 into 9 we have

$$Z_\sigma = \frac{B}{\sqrt{\det A}} \exp\left[-\frac{i}{2}\int d^4x\, d^4y\, \left(-\frac{1}{2}\phi^2(x)\right) g\delta^{(4)}(x-y) \left(-\frac{1}{2}\phi^2(y)\right)\right] \tag{12}$$

$$= \frac{B}{\sqrt{\det A}} \exp\left[i\int d^4x\, \left(-\frac{g}{8}\phi^4(x)\right)\right] \tag{13}$$

The integrand is now just the last term in the original Lagrangian 1, which was removed when we replaced $\mathcal{L}$ by $\mathcal{L}'$. In other words, the functional integrals of $\mathcal{L}$ and $\mathcal{L}'$ are the same, so the transformation 2 doesn't change the dynamics. [Presumably the constant $\frac{B}{\sqrt{\det A}}$ gets removed by normalization.]

The Euler-Lagrange equation for σ is

$$\frac{d}{dt}\frac{\partial \mathcal{L}'}{\partial \dot{\sigma}} = 0 = \frac{\partial \mathcal{L}'}{\partial \sigma} = \frac{1}{g}\left(\sigma - \frac{g}{2}\phi^2\right) \tag{14}$$

so

$$\sigma = \frac{g}{2}\phi^2 \tag{15}$$

Some of the Feynman diagrams for ϕ^4 theory are given in L&B's Chapter 19 (Fig. 19.6). At second order, there are 8 factors of ϕ (from $\phi^4 \times \phi^4$) to be contracted. We can contract one ϕ with the creation operator $a^\dagger$ and one at the other end with the annihilation operator a to get 2 external lines, leaving 6 ϕs to generate internal edges or loops.

I'm not sure what the Feynman diagrams for 3 would look like. At second order, we would have terms containing σ^4, $\sigma^3\phi^2$ and $\sigma^2\phi^4$, so presumably we would have diagrams involving various contractions within these terms.

Since contractions always occur in pairs and any uncontracted term gives zero, the $\sigma^3\phi^2$ term will contribute nothing, and we're left with σ^4 and $\sigma^2\phi^4$.