PERTURBATION EXPANSION FOR FUNCTIONAL INTEGRAL - MOTIVATION

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 24.3.

Post date: 27 Aug 2019.

We would like to do the following integral

$$Z(J)=\int dx\; e^{-\frac{1}{2}Ax^2-\frac{\lambda}{4!}x^4+Jx} \tag{1}$$

The term causing the problem is the term in x^4. However, we can write the exponential of this term as a power series

$$e^{-\frac{\lambda}{4!}x^4}=\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{\lambda}{4!}x^4\right)^n \tag{2}$$

We now observe that

$$\frac{\partial Z}{\partial J}=\int dx\; xe^{-\frac{1}{2}Ax^2-\frac{\lambda}{4!}x^4+Jx} \tag{3}$$

$$\frac{\partial^2 Z}{\partial J^2}=\int dx\; x^2e^{-\frac{1}{2}Ax^2-\frac{\lambda}{4!}x^4+Jx} \tag{4}$$

and so on, with the general result

$$\frac{\partial^m Z}{\partial J^m}=\int dx\; x^me^{-\frac{1}{2}Ax^2-\frac{\lambda}{4!}x^4+Jx} \tag{5}$$

Therefore, we can replace the exponential in 1 by

$$Z(J)=\int dx\;\left[e^{-\frac{1}{2}Ax^2+Jx}\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{\lambda}{4!}x^4\right)^n\right] \tag{6}$$

$$=\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{\lambda}{4!}\frac{\partial^4}{\partial J^4}\right)^n\int dx\; e^{-\frac{1}{2}Ax^2+Jx} \tag{7}$$

We can now do the integral as it's a standard Gaussian integral. We get

$$\int dx\; e^{-\frac{1}{2}Ax^2+Jx}=\sqrt{\frac{2\pi}{A}}e^{J^2/2A} \tag{8}$$

The series in 7 can be written as the exponential of an operator:

$$\sum_{n=0}^{\infty}\frac{1}{n!}\left(-\frac{\lambda}{4!}\frac{\partial^4}{\partial J^4}\right)^n = e^{-\frac{\lambda}{4!}\frac{\partial^4}{\partial J^4}} \tag{9}$$

so the final result is

$$Z(J) = \sqrt{\frac{2\pi}{A}}e^{-\frac{\lambda}{4!}\frac{\partial^4}{\partial J^4}}e^{J^2/2A} \tag{10}$$