GENERATING FUNCTIONAL FOR PHI-4 THEORY

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 24.4.

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The generating functional for ϕ^4 theory is

$$Z[J]=\left[\exp\left(-\frac{i\lambda}{4!}\int d^4z\frac{\delta^4}{\delta J(z)^4}\right)\mathcal{Z}_0[J]\right] \tag{1}$$

where the exponential is an expansion of the operator that forms the exponent, and the derivatives within this exponent are functional derivatives, and $\mathcal{Z}_0[J]$ is the normalized generating functional for the free scalar field:

$$\mathcal{Z}_0[J]=\exp\left(-\frac{1}{2}\int d^4x\, d^4y\; J(x)\Delta(x-y)J(y)\right) \tag{2}$$

The first term (apart from the trivial zeroth order term, which is 1) in the expansion of 1 is

$$Z_1[J]=-\frac{i\lambda}{4!}\int d^4z\frac{\delta^4}{\delta J(z)^4}\exp\left(-\frac{1}{2}\int d^4x\, d^4y\; J(x)\Delta(x-y)J(y)\right) \tag{3}$$

To calculate the first-order functional derivative, we use the original definition of a functional derivative

$$\frac{\delta F[f]}{\delta f(x_0)}\equiv\lim_{\epsilon\to0}\frac{F[f(x)+\epsilon\delta(x-x_0)]-F[f(x)]}{\epsilon} \tag{4}$$

In our case, the functional F is $\mathcal{Z}_0[J]$ and the function f with respect to which we're taking the functional derivative is J. The first functional derivative is therefore

$$\frac{\delta\mathcal{Z}_0[J]}{\delta J(z)}=\lim_{\epsilon\to0}\frac{1}{\epsilon}\left\{\exp\left(-\frac{1}{2}\int d^4x\, d^4y\;[J(x)+\epsilon\delta(z-x)]\Delta(x-y)[J(y)+\epsilon\delta(z-y)]\right)-\right. \tag{5}$$

$$\left.\exp\left(-\frac{1}{2}\int d^4x\, d^4y\; J(x)\Delta(x-y)J(y)\right)\right\}$$

Since ϵ is infinitesimal, we need worry only about terms up to first order in ϵ, so we can write the first exponential as

$$\exp\left(-\frac{1}{2}\int d^4x\, d^4y\ [J(x)+\epsilon\delta(z-x)]\,\Delta(x-y)\,[J(y)+\epsilon\delta(z-y)]\right) = \tag{6}$$

$$\exp\left(-\frac{1}{2}\int d^4x\, d^4y\ J(x)\,\Delta(x-y)\,J(y)\right)\times$$
$$\exp\left(-\frac{1}{2}\int d^4x\, d^4y\ J(x)\,\Delta(x-y)\,\epsilon\delta(z-y)\right)\times$$
$$\exp\left(-\frac{1}{2}\int d^4x\, d^4y\ \epsilon\delta(z-x)\,\Delta(x-y)\,J(y)\right) =$$
$$\mathcal{Z}_0[J]\exp\left(-\frac{1}{2}\int d^4x\, d^4y\ J(x)\,\Delta(x-y)\,\epsilon\delta(z-y)\right) \tag{7}$$
$$\exp\left(-\frac{1}{2}\int d^4x\, d^4y\ \epsilon\delta(z-x)\,\Delta(x-y)\,J(y)\right) \tag{8}$$

We can do the integrals over the delta functions in the last two lines to give, for these last two lines:

$$\mathcal{Z}_0[J]\exp\left(-\frac{1}{2}\int d^4x\ \ J(x)\,\Delta(x-z)\,\epsilon-\frac{1}{2}\int d^4y\ \epsilon\Delta(z-y)\,J(y)\right) \tag{9}$$

This exponential can be expanded up to first order to give

$$\mathcal{Z}_0[J]\left[1-\frac{1}{2}\int d^4x\ \ J(x)\,\Delta(x-z)\,\epsilon-\frac{1}{2}\int d^4y\ \epsilon\Delta(z-y)\,J(y)\right] \tag{10}$$

Since the propagator $\Delta(x-y)$ is symmetric ($\Delta(x-y)=\Delta(y-x)$ - see L&B's equation 17.24 and do the substitution $p\rightarrow -p$ in the integral), and the x in the first integral and the y in the second integral are just dummy integration variables, we have

$$\mathcal{Z}_0[J]\left[1-\frac{1}{2}\int d^4x\ \ J(x)\,\Delta(x-z)\,\epsilon-\frac{1}{2}\int d^4y\ \epsilon\Delta(z-y)\,J(y)\right] = \tag{11}$$
$$\mathcal{Z}_0[J]\left[1-\epsilon\int d^4y\ \Delta(z-y)\,J(y)\right]$$

Plugging this back into 5 we have

$$\frac{\delta \mathcal{Z}_0[J]}{\delta J(z)} = \left(-\int d^4y\, \Delta(z-y)\, J(y)\right) \mathcal{Z}_0[J] \tag{12}$$

To calculate the next derivative, we use the product rule, but then we need the derivative of the first factor in 12. Applying the same recipe 4 to this term gives

$$\frac{\delta}{\delta J(z)}\left(-\int d^4y\, \Delta(z-y)\, J(y)\right) = -\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int d^4y\, \Delta(z-y)\left[J(y)+\epsilon\delta(z-y)-J(y)\right] \tag{13}$$

$$= -\lim_{\epsilon\to 0}\frac{1}{\epsilon}\int d^4y\, \Delta(z-y)\,\epsilon\delta(z-y) \tag{14}$$

$$= -\Delta(z-z) \tag{15}$$

Therefore, the derivative of 12 is

$$\frac{\delta^2 \mathcal{Z}_0[J]}{\delta J(z)^2} = -\Delta(z-z)\,\mathcal{Z}_0[J] - \int d^4y\, \Delta(z-y)\, J(y)\,\frac{\delta \mathcal{Z}_0[J]}{\delta J(z)} \tag{16}$$

$$= -\Delta(z-z)\,\mathcal{Z}_0[J] - \int d^4y\, \Delta(z-y)\, J(y) \times \left(-\int d^4y\, \Delta(z-y)\, J(y)\right)\mathcal{Z}_0[J] \tag{17}$$

$$= \left[-\Delta(z-z) + \left(\int d^4y\, \Delta(z-y)\, J(y)\right)^2\right]\mathcal{Z}_0[J] \tag{18}$$

The next two derivatives are straighforward, as they don't involve the functional derivative of anything that we haven't already calculated. I'll use the shorthand notation

$$Y \equiv \int d^4y\, \Delta(z-y)\, J(y) \tag{19}$$

In this notation

$$\frac{\delta \mathcal{Z}_0[J]}{\delta J(z)} = -Y\,\mathcal{Z}_0[J] \tag{20}$$

$$\frac{\delta^2 \mathcal{Z}_0[J]}{\delta J(z)^2} = \left(-\Delta(z-z)+Y^2\right)\mathcal{Z}_0[J] \tag{21}$$

$$\frac{\delta Y}{\delta J(z)} = \Delta(z-z) \tag{22}$$

Applying the product and chain rules to 21 we have

$$\frac{\delta^3 \mathcal{Z}_0[J]}{\delta J(z)^3} = 2Y \frac{\delta Y}{\delta J(z)} \mathcal{Z}_0[J] + \left(-\Delta(z-z) + Y^2\right) \frac{\delta \mathcal{Z}_0[J]}{\delta J(z)} \tag{23}$$

$$= 2Y\Delta(z-z)\mathcal{Z}_0[J] - \left(-\Delta(z-z) + Y^2\right) Y \mathcal{Z}_0[J] \tag{24}$$

$$= 3Y\Delta(z-z)\mathcal{Z}_0[J] - Y^3 \mathcal{Z}_0[J] \tag{25}$$

$$= 3\Delta(z-z)\left(\int d^4y\, \Delta(z-y) J(y)\right) \mathcal{Z}_0[J] - \left(\int d^4y\, \Delta(z-y) J(y)\right)^3 \mathcal{Z}_0[J] \tag{26}$$

Finally, the fourth derivative can be found by differentiating 25:

$$\frac{\delta^4 \mathcal{Z}_0[J]}{\delta J(z)^4} = 3\Delta(z-z)^2 \mathcal{Z}_0[J] - 3Y^2\Delta(z-z)\mathcal{Z}_0[J] - \left(3Y\Delta(z-z) - Y^3\right) Y \mathcal{Z}_0[J] \tag{27}$$

$$= 3\Delta(z-z)^2 \mathcal{Z}_0[J] - 6Y^2\Delta(z-z)\mathcal{Z}_0[J] + Y^4 \mathcal{Z}_0[J] \tag{28}$$

$$= \left(3\Delta(z-z)^2 - 6Y^2\Delta(z-z) + Y^4\right) \mathcal{Z}_0[J] \tag{29}$$

The last line expands to L&B's equation 24.39 when we substitute for Y.

To get the final formula for $Z_1[J]$ from 3, we multiply $\frac{\delta^4 \mathcal{Z}_0[J]}{\delta J(z)^4}$ by $-\frac{i\lambda}{4!}$ and integrate over z. The only catch is that when we expand the powers of Y, we must use a different integration variable for each integral in the product. That is, for example

$$Y^2 = \left(\int d^4y\, \Delta(z-y) J(y)\right)^2 \tag{30}$$

$$= \int d^4y_1 d^4y_2\, \Delta(z-y_1) J(y_1) \Delta(z-y_2) J(y_2) \tag{31}$$

and similarly for Y^4. The final result is that given in L&B's equation 24.40, which is long, but follows directly from the above:

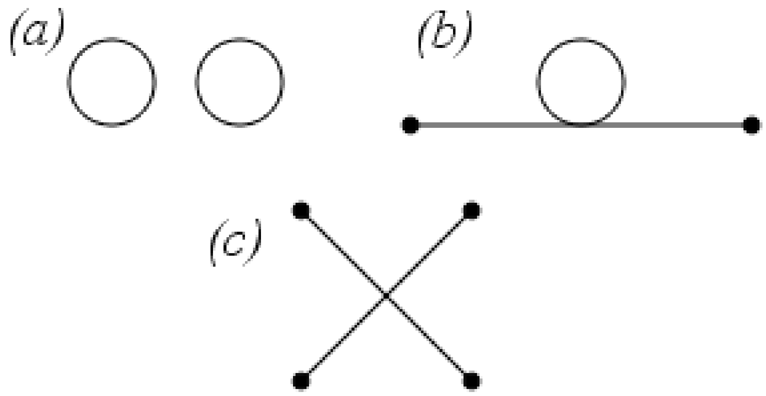

FIGURE 1. Feynman diagrams for $\mathcal{Z}_1[J]$

$$
\begin{aligned}
Z_1[J] = -i\lambda \Bigg[& \frac{1}{8}\int d^4z\,\Delta(z-z)^2 - \\
& \frac{1}{4}\int d^4z\,d^4y_1 d^4y_2\,\Delta(z-z)\,\Delta(z-y_1)\,J(y_1)\,\Delta(z-y_2)\,J(y_2) + \\
& \frac{1}{4!}\int d^4z\,d^4y_1 d^4y_2 d^4y_3 d^4y_4\,\Delta(z-y_1)\,J(y_1)\,\Delta(z-y_2)\,J(y_2)\times \\
& \Delta(z-y_3)\,J(y_3)\,\Delta(z-y_4)\,J(y_4)\Bigg]\mathcal{Z}_0[J]
\end{aligned} \tag{32}
$$

This is the sum of 3 terms, of order 0, 2 and 4 in J. If we interpret Δ as the Feynman propagator and each J as a vertex in the Feynman diagram, the diagrams for these three terms are as in Fig. 1. The propagator $\Delta(z-z)$ starts and ends at the same spacetime point, so is represented by a loop as in Fig. 1(a). The second term contains one of these loops and also two lines from the interaction point z to the endpoints y_1 and y_2 as in Fig. 1(b). Finally, the fourth order term has no loops and four branches, all meeting at a common point z, which is as a result of the ϕ^4 interaction term (as in Fig. 1(c)).