The generating functional for $\phi^4$ theory is

$$Z[J] = \exp \left( -\frac{i\lambda}{4!} \int d^4z \frac{\delta^4}{\delta J(z)^4} \right) Z_0[J]$$ (1)

where the exponential is an expansion of the operator that forms the exponent, and the derivatives within this exponent are functional derivatives, and $Z_0[J]$ is the normalized generating functional for the free scalar field:

$$Z_0[J] = \exp \left( -\frac{1}{2} \int d^4x d^4y J(x) \Delta(x-y) J(y) \right)$$ (2)

The first term (apart from the trivial zeroth order term, which is 1) in the expansion of $\delta Z$ is

$$Z_1[J] = -\frac{i\lambda}{4!} \int d^4z \frac{\delta^4}{\delta J(z)^4} \exp \left( -\frac{1}{2} \int d^4x d^4y J(x) \Delta(x-y) J(y) \right)$$ (3)

To calculate the first-order functional derivative, we use the original definition of a functional derivative

$$\frac{\delta F[f]}{\delta f(x_0)} \equiv \lim_{\epsilon \to 0} \frac{F[f(x) + \epsilon \delta(x-x_0)] - F[f(x)]}{\epsilon}$$ (4)

In our case, the functional $F$ is $Z_0[J]$ and the function $f$ with respect to which we’re taking the functional derivative is $J$. The first functional derivative is therefore

$$\frac{\delta Z_0[J]}{\delta J(z)} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left\{ \exp \left( -\frac{1}{2} \int d^4x d^4y \ [J(x) + \epsilon \delta(z-x)] \Delta(x-y) \ [J(y) + \epsilon \delta(z-y)] \right) - \exp \left( -\frac{1}{2} \int d^4x d^4y J(x) \Delta(x-y) J(y) \right) \right\}$$ (5)
Since \( \epsilon \) is infinitesimal, we need worry only about terms up to first order in \( \epsilon \), so we can write the first exponential as

\[
\exp \left( -\frac{1}{2} \int d^4x \, d^4y \, [J(x) + \epsilon \delta(z-x)] \Delta(x-y) [J(y) + \epsilon \delta(z-y)] \right) = 
\]

\[
\exp \left( -\frac{1}{2} \int d^4x \, d^4y \, J(x) \Delta(x-y) J(y) \right) \times \exp \left( -\frac{1}{2} \int d^4x \, d^4y \, J(x) \Delta(x-y) \epsilon \delta(z-y) \right) \times 
\]

\[
\exp \left( -\frac{1}{2} \int d^4x \, d^4y \, \epsilon \delta(z-x) \Delta(x-y) J(y) \right) = 
\]

\[
\mathcal{Z}_0[J] \exp \left( -\frac{1}{2} \int d^4x \, d^4y \, J(x) \Delta(x-y) \epsilon \delta(z-y) \right) 
\]

\[
\exp \left( -\frac{1}{2} \int d^4x \, d^4y \, \epsilon \delta(z-x) \Delta(x-y) J(y) \right) 
\]

\[
\mathcal{Z}_0[J] \left[ 1 - \frac{1}{2} \int d^4x \, J(x) \Delta(x-z) \epsilon \epsilon - \frac{1}{2} \int d^4y \, \epsilon \Delta(z-y) J(y) \right] = 
\]

Since the propagator \( \Delta(x-y) \) is symmetric (\( \Delta(x-y) = \Delta(y-x) \)) - see L&B’s equation 17.24 and do the substitution \( p \rightarrow -p \) in the integral, and the \( x \) in the first integral and the \( y \) in the second integral are just dummy integration variables, we have

\[
\mathcal{Z}_0[J] \left[ 1 - \frac{1}{2} \int d^4x \, J(x) \Delta(x-z) \epsilon \epsilon - \frac{1}{2} \int d^4y \, \epsilon \Delta(z-y) J(y) \right] = 
\]

\[
\mathcal{Z}_0[J] \left[ 1 - \epsilon \int d^4y \, \Delta(z-y) J(y) \right] = 
\]

Plugging this back into \( \mathcal{Z} \), we have
\[
\frac{\delta Z_0[J]}{\delta J(z)} = \left( - \int d^4y \: \Delta(z - y) \: J(y) \right) Z_0[J] \quad (12)
\]

To calculate the next derivative, we use the product rule, but then we need the derivative of the first factor in (12). Applying the same recipe to this term gives

\[
\frac{\delta}{\delta J(z)} \left( - \int d^4y \Delta(z - y) J(y) \right) = - \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int d^4y \Delta(z - y) [J(y) + \epsilon \delta(z - y) - J(y)]
\]

\[
= - \lim_{\epsilon \to 0} \frac{1}{\epsilon} \int d^4y \Delta(z - y) \epsilon \delta(z - y)
\]

\[
= - \Delta(z - z) \quad (14)
\]

Therefore, the derivative of (12) is

\[
\frac{\delta^2 Z_0[J]}{\delta J(z)^2} = - \Delta(z - z) \: Z_0[J] - \int d^4y \Delta(z - y) \: J(y) \: \frac{\delta Z_0[J]}{\delta J(z)}
\]

\[
= - \Delta(z - z) \: Z_0[J] - \int d^4y \Delta(z - y) \: J(y) \times \left( - \int d^4y \Delta(z - y) \: J(y) \right) Z_0[J]
\]

\[
= \left[ - \Delta(z - z) + \left( \int d^4y \Delta(z - y) \: J(y) \right)^2 \right] Z_0[J] \quad (18)
\]

The next two derivatives are straightforward, as they don’t involve the functional derivative of anything that we haven’t already calculated. I’ll use the shorthand notation

\[
Y \equiv \int d^4y \Delta(z - y) J(y)
\]

In this notation

\[
\frac{\delta Z_0[J]}{\delta J(z)} = - Y \: Z_0[J] \quad (20)
\]

\[
\frac{\delta^2 Z_0[J]}{\delta J(z)^2} = \left( - \Delta(z - z) + Y^2 \right) Z_0[J] \quad (21)
\]

\[
\frac{\delta Y}{\delta J(z)} = \Delta(z - z) \quad (22)
\]

Applying the product and chain rules to (21) we have
\[
\frac{\delta^3 Z_0 [J]}{\delta J (z)^3} = 2Y \frac{\delta Y}{\delta J (z)} Z_0 [J] + (-\Delta(z-z) + Y^2) \frac{\delta Z_0 [J]}{\delta J (z)} 
\]
\[
= 2Y \Delta(z-z) Z_0 [J] - (-\Delta(z-z) + Y^2) Y Z_0 [J] 
\]
\[
= 3Y \Delta(z-z) Z_0 [J] - Y^3 Z_0 [J] 
\]
\[
= 3\Delta(z-z) \left( \int d^4 y \Delta(z-y) J(y) \right) Z_0 [J] - \left( \int d^4 y \Delta(z-y) J(y) \right)^3 Z_0 [J] 
\]

Finally, the fourth derivative can be found by differentiating (23):

\[
\frac{\delta^4 Z_0 [J]}{\delta J (z)^4} = 3\Delta(z-z)^2 Z_0 [J] - 3Y^2 \Delta(z-z) Z_0 [J] - 
\]
\[
(3Y \Delta(z-z) - Y^3) Y Z_0 [J] 
\]
\[
= 3\Delta(z-z)^2 Z_0 [J] - 6Y^2 \Delta(z-z) Z_0 [J] + Y^4 Z_0 [J] 
\]
\[
= \left( 3\Delta(z-z)^2 - 6Y^2 \Delta(z-z) + Y^4 \right) Z_0 [J] 
\]

The last line expands to L&B’s equation 24.39 when we substitute for \( Y \).

To get the final formula for \( Z_1 [J] \) from (3), we multiply \( \frac{\delta^4 Z_0 [J]}{\delta J (z)^4} \) by \( -\frac{i}{4!} \) and integrate over \( z \). The only catch is that when we expand the powers of \( Y \), we must use a different integration variable for each integral in the product. That is, for example

\[
Y^2 = \left( \int d^4 y \Delta(z-y) J(y) \right)^2 
\]
\[
= \int d^4 y_1 d^4 y_2 \Delta(z-y_1) J(y_1) \Delta(z-y_2) J(y_2) 
\]

and similarly for \( Y^4 \). The final result is that given in L&B’s equation 24.40, which is long, but follows directly from the above:
$Z_1[J] = -i\lambda \left[ \frac{1}{8} \int d^4z \Delta(z-z)^2 - \right.$

$$\left. \frac{1}{4} \int d^4z d^4y_1 d^4y_2 \Delta(z-z)\Delta(z-y_1)J(y_1)\Delta(z-y_2)J(y_2) + \right.$$

$$\left. \frac{1}{4!} \int d^4z d^4y_1 d^4y_2 d^4y_3 d^4y_4 \Delta(z-y_1)J(y_1)\Delta(z-y_2)J(y_2) \times \Delta(z-y_3)J(y_3)\Delta(z-y_4)J(y_4) \right] Z_0[J]$$

This is the sum of 3 terms, of order 0, 2 and 4 in $J$. If we interpret $\Delta$ as the Feynman propagator and each $J$ as a vertex in the Feynman diagram, the diagrams for these three terms are as in Fig. [1]. The propagator $\Delta(z-z)$ starts and ends at the same spacetime point, so is represented by a loop as in Fig. [1](a). The second term contains one of these loops and also two lines from the interaction point $z$ to the endpoints $y_1$ and $y_2$ as in Fig. [1](b). Finally, the fourth order term has no loops and four branches, all meeting at a common point $z$, which is as a result of the $\phi^4$ interaction term (as in Fig. [1](c)).