TEMPERATURE DEPENDENCE OF PRESSURE IN PHI-4 THEORY

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 25.3.

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In L&B's example 25.4, they derive expressions for the partition function, energy and pressure of a statistical system in free scalar field theory. Their derivation skims over a few points that are worth clarifying in order that we can do their problem 25.3.

For free field theory (not ϕ^4 theory yet!), the partition function is given by their equation 25.35:

$$\ln Z=\mathcal{V}\int\frac{d^3p}{(2\pi)^3}\left[-\frac{1}{2}\beta E_{\boldsymbol{p}}-\ln\left(1-e^{-\beta E_{\boldsymbol{p}}}\right)\right] \tag{1}$$

where $\beta=\frac{1}{k_BT}$, with k_B the Boltzmann constant and T the temperature. From here, they derive the Helmholtz energy F. Although I've covered the Helmholtz energy in thermodynamcs, I haven't yet done any posts on statistical mechanics, so the formula in which F is derived from the partition function that L&B assume the reader knows at this point needs to be spelled out. It can be found in any textbook on statistical mechanics, and is

$$F=-\frac{1}{\beta}\ln Z \tag{2}$$

This gives us the formula in Example 25.4:

$$F=\frac{\mathcal{V}}{\beta}\int\frac{d^3p}{(2\pi)^3}\left[\frac{1}{2}\beta E_{\boldsymbol{p}}+\ln\left(1-e^{-\beta E_{\boldsymbol{p}}}\right)\right] \tag{3}$$

Next, they state the ground state energy, which they derive from the formula

$$E_0=-\frac{\partial}{\partial\beta}\ln Z \tag{4}$$

and which they just state as

$$E_0 = \frac{\mathcal{V}}{2}\int \frac{d^3p}{(2\pi)^3} E_{\boldsymbol{p}} \tag{5}$$

This requires a bit of explanation. Starting with 1, we take the derivative with respect to β to get

$$-\frac{\partial}{\partial\beta}\ln Z = \mathcal{V}\int \frac{d^3p}{(2\pi)^3}\left[\frac{1}{2}E_{\boldsymbol{p}} + \frac{E_{\boldsymbol{p}}e^{-\beta E_{\boldsymbol{p}}}}{1-e^{-\beta E_{\boldsymbol{p}}}}\right] \tag{6}$$

$$= \mathcal{V}\int \frac{d^3p}{(2\pi)^3}\frac{E_{\boldsymbol{p}} - E_{\boldsymbol{p}}e^{-\beta E_{\boldsymbol{p}}} + 2E_{\boldsymbol{p}}e^{-\beta E_{\boldsymbol{p}}}}{2\left(1-e^{-\beta E_{\boldsymbol{p}}}\right)} \tag{7}$$

$$= \frac{\mathcal{V}}{2}\int \frac{d^3p}{(2\pi)^3}E_{\boldsymbol{p}}\frac{1+e^{-\beta E_{\boldsymbol{p}}}}{1-e^{-\beta E_{\boldsymbol{p}}}} \tag{8}$$

We're after the ground state energy, which is the energy as the temperature $T \to 0$. In this case $\beta = \frac{1}{k_B T} \to \infty$ and $e^{-\beta E_{\boldsymbol{p}}} \to 0$, so we're left with

$$E_0 = \lim_{T\to 0}\frac{\mathcal{V}}{2}\int \frac{d^3p}{(2\pi)^3}E_{\boldsymbol{p}}\frac{1+e^{-\beta E_{\boldsymbol{p}}}}{1-e^{-\beta E_{\boldsymbol{p}}}} \tag{9}$$

$$= \frac{\mathcal{V}}{2}\int \frac{d^3p}{(2\pi)^3}E_{\boldsymbol{p}} \tag{10}$$

which agrees with 5. The energy at $T \neq 0$ is then given relative to the ground state energy, and this can be found by subtracting 5 from 6 which gives

$$E - E_0 = \mathcal{V}\int \frac{d^3p}{(2\pi)^3}\frac{E_{\boldsymbol{p}}e^{-\beta E_{\boldsymbol{p}}}}{1-e^{-\beta E_{\boldsymbol{p}}}} \tag{11}$$

$$= \mathcal{V}\int \frac{d^3p}{(2\pi)^3}\frac{E_{\boldsymbol{p}}}{e^{\beta E_{\boldsymbol{p}}}-1} \tag{12}$$

which gives L&B's equation 25.36.

The ground state pressure P_0 is given by the thermodynamic expression

$$P_0 = -\left.\frac{\partial F}{\partial\mathcal{V}}\right|_{T=0} = \frac{E_0}{\mathcal{V}} = \frac{1}{2}\int \frac{d^3p}{(2\pi)^3}E_{\boldsymbol{p}} \tag{13}$$

The pressure relative to the ground state pressure is then given from 3 and 13:

$$P - P_0 = -\frac{\partial F}{\partial \mathcal{V}} + \left.\frac{\partial F}{\partial \mathcal{V}}\right|_{T=0} \tag{14}$$

$$= \frac{\mathcal{V}}{\beta} \int \frac{d^3p}{(2\pi)^3} \ln\left(1 - e^{-\beta E_{\boldsymbol{p}}}\right) \tag{15}$$

which gives L&B's equation 25.37.

We can now apply this to ϕ^4 theory, starting with L&B's equation 25.45 for the temperature-dependent part of $\ln Z$:

$$\ln Z_{(1)} = -\frac{\lambda\beta\mathcal{V}}{8} \left[\int \frac{d^3p}{(2\pi)^3} \frac{1}{E_{\boldsymbol{p}}} \left(\frac{1}{e^{\beta E_{\boldsymbol{p}}} - 1}\right)\right]^2 \tag{16}$$

In what follows, I'll deal only with this part of $\ln Z$.

We get

$$F = -\frac{1}{\beta} \ln Z \tag{17}$$

$$= \frac{\lambda\mathcal{V}}{8} \left[\int \frac{d^3p}{(2\pi)^3} \frac{1}{E_{\boldsymbol{p}}} \left(\frac{1}{e^{\beta E_{\boldsymbol{p}}} - 1}\right)\right]^2 \tag{18}$$

The pressure is

$$P = -\frac{\partial F}{\partial \mathcal{V}} \tag{19}$$

$$= -\frac{\lambda}{8} \left[\int \frac{d^3p}{(2\pi)^3} \frac{1}{E_{\boldsymbol{p}}} \left(\frac{1}{e^{\beta E_{\boldsymbol{p}}} - 1}\right)\right]^2 \tag{20}$$

As the mass $m \to 0$, $E_{\boldsymbol{p}} = \sqrt{\boldsymbol{p}^2 + m^2} \to p$, where p is the magnitude of the 3-momentum. We're therefore faced with the integral

$$I = \int \frac{d^3p}{(2\pi)^3} \frac{1}{p} \left(\frac{1}{e^{\beta p} - 1}\right) \tag{21}$$

$$= \frac{4\pi}{(2\pi)^3} \int_0^\infty p^2 dp \frac{1}{p} \left(\frac{1}{e^{\beta p} - 1}\right) \tag{22}$$

$$= \frac{1}{2\pi^2} \int_0^\infty dp \frac{p}{e^{\beta p} - 1} \tag{23}$$

I used Maple to do the integral, although I'd guess that it can be found in tables somewhere. The result is

$$I = \frac{1}{12\beta^2} = \frac{k_B^2 T^2}{12} \tag{24}$$

From 20 we get

$$P = -\frac{\lambda}{8}\frac{k_B^4 T^4}{144} \tag{25}$$

Thus the pressure has a T^4 dependence for $m \to 0$.

As an aside, if you're interested in the integral in 23, Maple gives it as

$$\int \frac{p}{e^{\beta p} - 1} dp = -\frac{p^2}{2} + \frac{p \ln\left(1 - \mathrm{e}^{\beta p}\right)}{\beta} + \frac{\mathit{polylog}\left(2, \mathrm{e}^{\beta p}\right)}{\beta^2} \tag{26}$$

The polylog is a generalization of the logarithm which is defined as a series:

$$polylog\,(n, x) = \sum_{k=1}^{\infty} \frac{x^k}{k^n} \tag{27}$$