WICK ROTATION OF THE SCHRÖDINGER EQUATION

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 25.4.

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A Wick rotation of the four-dimensional spacetime coordinate system amounts to replacing the time t (or x_0) by an imaginary time $-i\tau$ and the energy component p_0 of the momentum 4-vector by $i\omega$. As shown in section 25.1, this results in the spacetime metric being given by $(+,+,+,+)$ so spacetime becomes Euclidean rather than Minkowski.

The Schrödinger equation in Minkowski spacetime is

$$i\partial_0\psi = H\psi \tag{1}$$

After a Wick rotation

$$\partial_0 = \frac{\partial}{\partial t} \rightarrow \frac{\partial}{-i\partial\tau} = i\frac{\partial}{\partial\tau} \tag{2}$$

The Wick rotated Schrödinger equation is therefore

$$-\frac{\partial}{\partial\tau}\psi = H\psi \tag{3}$$

This assumes that the hamiltonian H does not depend explicitly on time.

The formal solution of this is

$$\psi(t) = \psi(0)\,e^{-H\tau} \tag{4}$$

The original Heisenberg equation of motion for an operator is derived from

$$A(t) = e^{-iHt}Ae^{iHt} \tag{5}$$

where A is the operator in the Schrödinger picture (time-independent operator) and $A(t)$ is the operator in the Heisenberg picture.

After a Wick rotation, the time dependence (in terms of τ) is given by

$$A(\tau) = e^{H\tau}Ae^{-H\tau} \tag{6}$$

Taking the derivative we get

$$\frac{\partial A(\tau)}{\partial \tau} = He^{H\tau}Ae^{-H\tau} - e^{H\tau}Ae^{-H\tau}H \tag{7}$$
$$= HA(\tau) - A(\tau)H \tag{8}$$
$$= [H, A(\tau)] \tag{9}$$

If the hamiltonian is

$$H = \omega c^{\dagger}c \tag{10}$$

we can find the time dependent versions of $c^{\dagger}$ and c. I'm assuming here that L&B mean these operators to be creation and annihilation operators for the harmonic oscillator, so that they satisfy the commutation relation

$$\left[c, c^{\dagger}\right] = 1 \tag{11}$$

from which we have

$$cc^{\dagger} = 1 + c^{\dagger}c \tag{12}$$
$$c^{\dagger}c = cc^{\dagger} - 1 \tag{13}$$

In that case, we have

$$\partial_{\tau}c(\tau) = [H, c] \tag{14}$$
$$= \omega\left(c^{\dagger}cc - cc^{\dagger}c\right) \tag{15}$$
$$= \omega\left(c^{\dagger}cc - c - c^{\dagger}cc\right) \tag{16}$$
$$= -\omega c \tag{17}$$

So we have

$$c(\tau) = c(0)e^{-\omega\tau} \tag{18}$$

For $c^{\dagger}$ we have

$$\partial_{\tau}c^{\dagger}(\tau) = \left[H, c^{\dagger}\right] \tag{19}$$
$$= \omega\left(c^{\dagger}cc^{\dagger} - c^{\dagger}c^{\dagger}c\right) \tag{20}$$
$$= \omega\left(c^{\dagger}cc^{\dagger} + c^{\dagger} - c^{\dagger}cc^{\dagger}\right) \tag{21}$$
$$= \omega c^{\dagger} \tag{22}$$

so we have

$$c^{\dagger}\left(\tau\right)=c^{\dagger}\left(0\right)e^{\omega\tau} \tag{23}$$