WICK ROTATION OF THE SCHRÖDINGER EQUATION

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A Wick rotation of the four-dimensional spacetime coordinate system amounts to replacing the time $t$ (or $x_0$) by an imaginary time $-i\tau$ and the energy component $p_0$ of the momentum 4-vector by $i\omega$. As shown in section 25.1, this results in the spacetime metric being given by $(+,+,+,+)$ so spacetime becomes Euclidean rather than Minkowski.

The Schrödinger equation in Minkowski spacetime is

$$i\partial_0\psi = H\psi$$

(1)

After a Wick rotation

$$\partial_0 = \frac{\partial}{\partial t} \rightarrow \frac{\partial}{-i\partial\tau} = i \frac{\partial}{\partial\tau}$$

(2)

The Wick rotated Schrödinger equation is therefore

$$-\frac{\partial}{\partial\tau}\psi = H\psi$$

(3)

This assumes that the hamiltonian $H$ does not depend explicitly on time.

The formal solution of this is

$$\psi(t) = \psi(0) e^{-H\tau}$$

(4)

The original Heisenberg equation of motion for an operator is derived from

$$A(t) = e^{-iHt} Ae^{iHt}$$

(5)

where $A$ is the operator in the Schrödinger picture (time-independent operator) and $A(t)$ is the operator in the Heisenberg picture.

After a Wick rotation, the time dependence (in terms of $\tau$) is given by

$$A(\tau) = e^{H\tau} Ae^{-H\tau}$$

(6)

Taking the derivative we get
\[ \frac{\partial A(\tau)}{\partial \tau} = H e^{H\tau} A e^{-H\tau} - e^{H\tau} A e^{-H\tau} H \]  
(7)

\[ = H A(\tau) - A(\tau) H \]  
(8)

\[ = [H, A(\tau)] \]  
(9)

If the hamiltonian is

\[ H = \omega c^\dagger c \]  
(10)

we can find the time dependent versions of \( c^\dagger \) and \( c \). I’m assuming here that L&B mean these operators to be creation and annihilation operators for the harmonic oscillator, so that they satisfy the commutation relation

\[ [c, c^\dagger] = 1 \]  
(11)

from which we have

\[ cc^\dagger = 1 + c^\dagger c \]  
(12)

\[ c^\dagger c = cc^\dagger - 1 \]  
(13)

In that case, we have

\[ \partial_\tau c(\tau) = [H, c] \]  
(14)

\[ = \omega \left( c^\dagger cc - cc^\dagger c \right) \]  
(15)

\[ = \omega \left( c^\dagger cc - c - c^\dagger cc \right) \]  
(16)

\[ = -\omega c \]  
(17)

So we have

\[ c(\tau) = c(0) e^{-\omega \tau} \]  
(18)

For \( c^\dagger \) we have

\[ \partial_\tau c^\dagger(\tau) = \left[ H, c^\dagger \right] \]  
(19)

\[ = \omega \left( c^\dagger cc^\dagger - c^\dagger c^\dagger c \right) \]  
(20)

\[ = \omega \left( c^\dagger cc^\dagger + c^\dagger c - c^\dagger cc^\dagger \right) \]  
(21)

\[ = \omega c^\dagger \]  
(22)

so we have
\[ c^\dagger (\tau) = c^\dagger (0) e^{i\omega \tau} \] (23)