GOLDSTONE'S THEOREM

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 26.3.

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We start with a continuous symmetry with Noether charge

$$Q_N = \int d^3x\ J_N^0(x) \tag{1}$$

where J^μ is the Noether four-current. We assume that the vacuum is not invariant, so that

$$Q_N|0\rangle \neq 0 \tag{2}$$

We consider a field $\phi(x)$ that is not invariant under Q_N so that

$$[Q_N, \phi(x)] = \psi(x) \tag{3}$$

where ψ is some other field. We'll look at $\langle 0|\psi(0)|0\rangle$ which we assume takes on a non-zero value when symmetry is broken:

$$\langle 0|\psi(0)|0\rangle \neq 0 \tag{4}$$

First, we show that $\langle 0|\psi(0)|0\rangle$ is independent of time. Taking the time derivative, we have

$$\frac{\partial}{\partial x^0}\langle 0|\psi(0)|0\rangle = \partial_0\langle 0[Q_N, \phi(0)]0\rangle \tag{5}$$

$$= \partial_0\int d^3x\left\langle 0\left[J_N^0(x), \phi(0)\right]0\right\rangle \tag{6}$$

At this point, we impose the continuity condition

$$\partial_\mu J_N^\mu = 0 \tag{7}$$

so we have

$$\partial_0\langle 0|\psi(0)|0\rangle = -\int d^3x\langle 0[\nabla\cdot\boldsymbol{J}_N(x), \phi(0)]0\rangle \tag{8}$$

We can do the usual trick of transforming this volume integral of a divergence to a surface integral using Gauss's theorem, so we have

$$\partial_0 \langle 0 | \psi(0) | 0 \rangle = -\int d\boldsymbol{S} \cdot \langle 0 [\boldsymbol{J}_N(x), \phi(0)] 0 \rangle \tag{9}$$

where the integral is now over the surface $\boldsymbol{S}$ enclosing the volume. The usual argument is that this integral goes to zero as the surface goes to infinity, which relies on the integrand falling off to zero fast enough. The argument here, I think, is that we are taking the commutator of a field $\phi(0)$ at the origin with the Noether current $\boldsymbol{J}_N(x)$ at spatial points which become so distant from the origin that no light signal could connect the origin with these points. That is, the separation of the points x and 0 is spacelike, so they cannot influence each other, so any commutator of two functions separated by a spacelike interval must be zero.

Using the translation operator, we have

$$J_N^0(x) = e^{ip\cdot x} J_N^0(0) e^{-ip\cdot x} \tag{10}$$

Going back to 3, we have

$$\langle 0 | \psi(0) | 0 \rangle = \langle 0 | [Q_N, \phi(0)] | 0 \rangle \tag{11}$$

$$= \int d^3x \left\langle 0 \left| \left[e^{ip\cdot x} J_N^0(0) e^{-ip\cdot x}, \phi(0) \right] \right| 0 \right\rangle \tag{12}$$

$$= \int d^3x \left\langle 0 \left| e^{ip\cdot x} J_N^0(0) e^{-ip\cdot x} \phi(0) - \phi(0) e^{ip\cdot x} J_N^0(0) e^{-ip\cdot x} \right| 0 \right\rangle \tag{13}$$

We can now insert a resolution of the identity in the form

$$1 = \sum_n |n\rangle \langle n| \tag{14}$$

to get

$$\langle 0 | \psi(0) | 0 \rangle = \int d^3x \sum_n \left[\left\langle 0 \left| e^{ip\cdot x} J_N^0(0) e^{-ip\cdot x} \right| n \right\rangle \langle n | \phi(0) | 0 \rangle - \langle 0 | \phi(0) | n \rangle \left\langle n \left| e^{ip\cdot x} J_N^0(0) e^{-ip\cdot x} \right| 0 \right\rangle \right] \tag{15}$$

$$= \int d^3x \sum_n \left[\left\langle 0 \left| J_N^0(0) \right| n \right\rangle \langle n | \phi(0) | 0 \rangle e^{-ip_n\cdot x} - \langle 0 | \phi(0) | n \rangle \left\langle n \left| J_N^0(0) \right| 0 \right\rangle e^{ip_n\cdot x} \right] \tag{16}$$

Here we've used the fact that the state $|n\rangle$ is a state with a given momentum p_n, so the operator $e^{-ip\cdot x}$ operating on this state gives $e^{-ip_n\cdot x}|n\rangle$. The four-momentum operator acting on the vacuum state $|0\rangle$ gives 0. By the assumption 4, this expression is non-zero.

The spatial integral can now be done using the identity

$$\int d^3x\, e^{\pm i\boldsymbol{p}_n\cdot\boldsymbol{x}} = (2\pi)^3\,\delta^{(3)}(\boldsymbol{p}_n) \tag{17}$$

Doing this integral, we get

In eqn 26.31 in L&B, the ψ fields should be ϕ. I also get the opposite sign for the two exponentials, but this doesn't affect the derivation.

$$\langle 0|\psi(0)|0\rangle = (2\pi)^3 \sum_n \delta^{(3)}(\boldsymbol{p}_n)\left[\left\langle 0\left|J_N^0(0)\right|n\right\rangle \langle n|\phi(0)|0\rangle\, e^{-ip_n^0\cdot x^0} - \right.$$
$$\left. \langle 0|\phi(0)|n\rangle \left\langle n\left|J_N^0(0)\right|0\right\rangle e^{ip_n^0\cdot x^0}\right] \tag{18}$$

From the above argument, this expression is independent of time, and is non-zero.

First, we can observe that the term with $n=0$ is zero, since the vacuum state has zero energy, so $p_0^0=0$, so we have

$$\langle 0|\psi(0)|0\rangle_0 = \left\langle 0\left|J_N^0(0)\right|0\right\rangle \langle n|\phi(0)|0\rangle - $$
$$\langle 0|\phi(0)|0\rangle \left\langle 0\left|J_N^0(0)\right|0\right\rangle \tag{19}$$
$$= 0 \tag{20}$$

If we take the time derivative of 18, we should get zero. We get

$$\partial_0 \langle 0|\psi(0)|0\rangle = (2\pi)^3\, i \sum_n p_n^0 \delta^{(3)}(\boldsymbol{p}_n)\left[-\left\langle 0\left|J_N^0(0)\right|n\right\rangle \langle n|\phi(0)|0\rangle\, e^{-ip_n^0\cdot x^0} - \right.$$
$$\left. \langle 0|\phi(0)|n\rangle \left\langle n\left|J_N^0(0)\right|0\right\rangle e^{ip_n^0\cdot x^0}\right] \tag{21}$$

If the particle with four-momentum p_n is massive, then even when the 3-momentum $\boldsymbol{p}_n = 0$ (as required by the $\delta^{(3)}(\boldsymbol{p}_n)$), $p_n^0 = m \neq 0$. In this case, in order for $\partial_0\langle 0|\psi(0)|0\rangle = 0$ to be true, we must have

$$\langle 0|\phi(0)|n\rangle \left\langle n\left|J_N^0(0)\right|0\right\rangle = 0 \tag{22}$$

I'm not sure how we can deduce the more restrictive condition that $\left\langle n\left|J_N^0(0)\right|0\right\rangle = 0$ on its own. Is there some reason why $\langle 0|\phi(0)|n\rangle$ must be non-zero? Comments welcome.

In any case, the only situation we're left with is the case of the particle being massless. In this case, as $\boldsymbol{p}_n \to 0$, $p_n^0 \to 0$, and in this case, 21 can be zero even if $\left\langle n\left|J_N^0(0)\right|0\right\rangle \neq 0$. That is, any states linked to the ground state via the Noether current as massless, and are known as Goldstone modes.