SYMMETRY BREAKING IN A GAUGE THEORY

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 26.4.

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We consider the Lagrangian

$$\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+\left|D_{\mu}\psi\right|^{2}-V\left(\psi\right) \tag{1}$$

where $F_{\mu\nu}$ is the electromagnetic field tensor, D_{μ} is a covariant derivative defined by

$$D_{\mu}=\partial_{\mu}+iqA_{\mu} \tag{2}$$

with A_{μ}being the electromagnetic four-potential, and the potential term is given in terms of a complex scalar field ψ as

$$V\left(\psi\right)=-m^{2}\psi^{\dagger}\psi+\frac{\lambda}{2}\left(\psi^{\dagger}\psi\right)^{2} \tag{3}$$

Note that the Lagrangian is invariant under the symmetry transformation

$$\psi\to\psi e^{i\alpha(x)} \tag{4}$$

where $\alpha\left(x\right)$ is an arbitrary real function of spacetime x. This is shown in L&B's Chapter 14, particularly Example 14.3 (although beware this example has a few typos in it).

If we take the quantity $\psi^{\dagger}\psi$ to be a single variable called $a\equiv\psi^{\dagger}\psi$, we can calculate the minimum of V by taking the derivative.

$$\frac{\partial V\left(a\right)}{\partial a}=-m^{2}+\lambda a=0 \tag{5}$$

which gives us

$$a_{0}=\frac{m^{2}}{\lambda} \tag{6}$$

Thus the potential has a minimum whenever

$$\psi=\sqrt{\frac{m^{2}}{\lambda}}e^{i\alpha} \tag{7}$$

We can break the symmetry by choosing a particular value of ψ, which we'll call ψ_0. The simplest choice is

$$\psi_0 = \sqrt{\frac{m^2}{\lambda}} \tag{8}$$

Since ψ is a complex function of x, we can represent ψ near this minimum by the expression

$$\psi = \psi_0 + \frac{1}{\sqrt{2}}\left(\phi_1(x) + i\phi_2(x)\right) \tag{9}$$

$$\psi^\dagger = \psi_0 + \frac{1}{\sqrt{2}}\left(\phi_1(x) - i\phi_2(x)\right) \tag{10}$$

where ϕ_1 and ϕ_2 are taken to be small real functions.

The potential in this broken symmetry theory can then be found by plugging 9 into 3 and multiplying out the terms. We get

$$\psi^\dagger\psi = \psi_0^2 + \sqrt{2}\psi_0\phi_1 + \frac{1}{2}\left(\phi_1^2 + \phi_2^2\right) \tag{11}$$

$$= \frac{m^2}{\lambda} + \sqrt{2}\sqrt{\frac{m^2}{\lambda}}\phi_1 + \frac{1}{2}\left(\phi_1^2 + \phi_2^2\right) \tag{12}$$

$$\left(\psi^\dagger\psi\right)^2 = \psi_0^4 + 2\sqrt{2}\psi_0^3\phi_1 + 2\psi_0^2\phi_1^2 + \psi_0^2\left(\phi_1^2 + \phi_2^2\right) + \mathcal{O}\left(\phi_i^3\right) \tag{13}$$

$$= \frac{m^4}{\lambda^2} + 2\frac{m^2}{\lambda}\sqrt{\frac{2m^2}{\lambda}}\phi_1 + 2\frac{m^2}{\lambda}\phi_1^2 + \frac{m^2}{\lambda}\left(\phi_1^2 + \phi_2^2\right) + \mathcal{O}\left(\phi_i^3\right) \tag{14}$$

If we call this potential $U(x)$, we have

$$U(x) = -m^2\psi^\dagger\psi + \frac{\lambda}{2}\left(\psi^\dagger\psi\right)^2 = -\frac{m^4}{\lambda} - m^2\sqrt{\frac{2m^2}{\lambda}}\phi_1 - \frac{m^2}{2}\left(\phi_1^2 + \phi_2^2\right) + \frac{m^4}{2\lambda} + \frac{m^2}{\lambda}\lambda\sqrt{\frac{2m^2}{\lambda}}\phi_1 + m^2\phi_1^2 + \frac{m^2}{2}\left(\phi_1^2 + \phi_2^2\right) + \mathcal{O}\left(\phi_i^3\right) \tag{15}$$

$$= -\frac{m^4}{2\lambda} + m^2\phi_1^2 + \mathcal{O}\left(\phi_i^3\right) \tag{16}$$

The kinetic energy term becomes

$$|D_\mu\psi|^2 = \left(\partial_\mu\psi + iqA_\mu\psi\right)\left(\partial^\mu\psi^\dagger - iqA^\mu\psi^\dagger\right) \tag{17}$$

We can now insert 9 and multiply out the result. Remember that ψ_0 is a constant, so its derivative is zero.

$$\begin{aligned}
|D_\mu\psi|^2 &= \left[\frac{1}{\sqrt{2}}(\partial_\mu\phi_1 + i\partial_\mu\phi_2) + iqA_\mu\left(\psi_0 + \frac{1}{\sqrt{2}}(\phi_1 + i\phi_2)\right)\right]\times \\
&\quad \left[\frac{1}{\sqrt{2}}(\partial^\mu\phi_1 - i\partial^\mu\phi_2) - iqA^\mu\left(\psi_0 + \frac{1}{\sqrt{2}}(\phi_1 - i\phi_2)\right)\right] \qquad (18) \\
&= \frac{1}{2}(\partial_\mu\phi_1)^2 + \frac{1}{2}(\partial_\mu\phi_2)^2 + \frac{1}{\sqrt{2}}\partial_\mu\phi_1(-iqA^\mu\psi_0) + iqA_\mu\psi_0\frac{1}{\sqrt{2}}\partial^\mu\phi_1 + \\
&\quad \frac{i}{\sqrt{2}}\partial_\mu\phi_2(-iqA^\mu\psi_0) + \frac{i}{\sqrt{2}}qA_\mu\psi_0(-i\partial^\mu\phi_2) + q^2A^\mu A_\mu\psi_0^2 + \ldots \qquad (19) \\
&= \frac{1}{2}(\partial_\mu\phi_1)^2 + \frac{1}{2}(\partial_\mu\phi_2)^2 + \sqrt{2}q\psi_0 A_\mu\partial^\mu\phi_2 + q^2\psi_0^2A^\mu A_\mu + \ldots \qquad (20) \\
&= \frac{1}{2}(\partial_\mu\phi_1)^2 + \frac{1}{2}(\partial_\mu\phi_2)^2 + q\sqrt{\frac{2m^2}{\lambda}}A_\mu\partial^\mu\phi_2 + \frac{q^2}{\lambda}m^2A^\mu A_\mu + \ldots \qquad (21)
\end{aligned}$$

The term $\frac{q^2}{\lambda}m^2A^\mu A_\mu$ resembles a term with a mass governed by the field A^μ which is the electromagnetic field, so I suppose this term could be interpreted as a massive photon.