SYMMETRY BREAKING IN A GAUGE THEORY

We consider the Lagrangian

\[ \mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + |D_\mu \psi|^2 - V(\psi) \]  

(1)

where \( F_{\mu\nu} \) is the electromagnetic field tensor, \( D_\mu \) is a covariant derivative defined by

\[ D_\mu = \partial_\mu + iqA_\mu \]  

(2)

with \( A_\mu \) being the electromagnetic four-potential, and the potential term is given in terms of a complex scalar field \( \psi \) as

\[ V(\psi) = -m^2 \psi^\dagger \psi + \frac{\lambda}{2} (\psi^\dagger \psi)^2 \]  

(3)

Note that the Lagrangian is invariant under the symmetry transformation

\[ \psi \rightarrow \psi e^{i\alpha(x)} \]  

(4)

where \( \alpha(x) \) is an arbitrary real function of spacetime \( x \). This is shown in L&B’s Chapter 14, particularly Example 14.3 (although beware this example has a few typos in it).

If we take the quantity \( \psi^\dagger \psi \) to be a single variable called \( a \equiv \psi^\dagger \psi \), we can calculate the minimum of \( V \) by taking the derivative.

\[ \frac{\partial V(a)}{\partial a} = -m^2 + \lambda a = 0 \]  

(5)

which gives us

\[ a_0 = \frac{m^2}{\lambda} \]  

(6)

Thus the potential has a minimum whenever

\[ \psi = \sqrt{\frac{m^2}{\lambda}} e^{i\alpha} \]  

(7)
We can break the symmetry by choosing a particular value of $\psi$, which we’ll call $\psi_0$. The simplest choice is

$$\psi_0 = \sqrt{\frac{m^2}{\lambda}} \quad (8)$$

Since $\psi$ is a complex function of $x$, we can represent $\psi$ near this minimum by the expression

$$\psi = \psi_0 + \frac{1}{\sqrt{2}} \left( \phi_1(x) + i\phi_2(x) \right) \quad (9)$$

$$\psi^\dagger = \psi_0 + \frac{1}{\sqrt{2}} \left( \phi_1(x) - i\phi_2(x) \right) \quad (10)$$

where $\phi_1$ and $\phi_2$ are taken to be small real functions.

The potential in this broken symmetry theory can then be found by plugging (9) into (3) and multiplying out the terms. We get

$$\psi^\dagger \psi = \psi_0^2 + \frac{1}{\sqrt{2}} \psi_0 \phi_1 + \frac{1}{2} \left( \phi_1^2 + \phi_2^2 \right) \quad (11)$$

$$= \frac{m^2}{\lambda} + \sqrt{\frac{m^2}{\lambda}} \phi_1 + \frac{1}{2} \left( \phi_1^2 + \phi_2^2 \right) \quad (12)$$

$$\left( \psi^\dagger \psi \right)^2 = \psi_0^4 + 2\sqrt{2} \psi_0^3 \phi_1 + 2\psi_0^2 \phi_1^2 + \psi_0^2 \left( \phi_1^2 + \phi_2^2 \right) + O \left( \phi_3^3 \right) \quad (13)$$

$$= \frac{m^4}{\lambda^2} + 2 \frac{m^2}{\lambda} \sqrt{2 \frac{m^2}{\lambda}} \phi_1 + 2 \frac{m^2}{\lambda} \phi_1^2 + \frac{m^2}{\lambda} \left( \phi_1^2 + \phi_2^2 \right) + O \left( \phi_3^3 \right) \quad (14)$$

If we call this potential $U(x)$, we have

$$U(x) = -m^2 \psi^\dagger \psi + \frac{\lambda}{2} \left( \psi^\dagger \psi \right)^2 = -\frac{m^4}{\lambda} - m^2 \sqrt{\frac{2m^2}{\lambda}} \phi_1 - \frac{m^2}{2} \left( \phi_1^2 + \phi_2^2 \right) +$$

$$\frac{m^4}{2\lambda} + \frac{m^2}{\lambda} \sqrt{\frac{2m^2}{\lambda}} \phi_1 + m^2 \phi_1^2 + \frac{m^2}{2} \left( \phi_1^2 + \phi_2^2 \right) + O \left( \phi_3^3 \right) \quad (15)$$

$$= -\frac{m^4}{2\lambda} + m^2 \phi_1^2 + O \left( \phi_3^3 \right) \quad (16)$$

The kinetic energy term becomes

$$|D_\mu \psi|^2 = (\partial_\mu \psi + iqA_\mu \psi) \left( \partial^\mu \psi^\dagger - iqA^\mu \psi^\dagger \right) \quad (17)$$
We can now insert $\psi_0$ and multiply out the result. Remember that $\psi_0$ is a constant, so its derivative is zero.

$$|D_\mu \psi|^2 = \left[ \frac{1}{\sqrt{2}} (\partial_\mu \phi_1 + i \partial_\mu \phi_2) + iqA_\mu \left( \psi_0 + \frac{1}{\sqrt{2}} (\phi_1 + i \phi_2) \right) \right] \times \left[ \frac{1}{\sqrt{2}} (\partial^\mu \phi_1 - i \partial^\mu \phi_2) - iqA^\mu \left( \psi_0 + \frac{1}{\sqrt{2}} (\phi_1 - i \phi_2) \right) \right]$$

(18)

$$= \frac{1}{2} (\partial_\mu \phi_1)^2 + \frac{1}{2} (\partial_\mu \phi_2)^2 + \frac{1}{\sqrt{2}} \partial_\mu \phi_1 (-iqA^\mu \psi_0) + iqA_\mu \psi_0 \frac{1}{\sqrt{2}} \partial^\mu \phi_1 + \frac{i}{\sqrt{2}} \partial_\mu \phi_2 (-iqA^\mu \psi_0) + \frac{i}{\sqrt{2}} qA_\mu \psi_0 (-i \partial^\mu \phi_2) + q^2 A^\mu A_\mu \psi_0^2 + \ldots$$

(19)

$$= \frac{1}{2} (\partial_\mu \phi_1)^2 + \frac{1}{2} (\partial_\mu \phi_2)^2 + \sqrt{2} q \psi_0 A_\mu \partial^\mu \phi_2 + q^2 \psi_0^2 A^\mu A_\mu + \ldots$$

(20)

$$= \frac{1}{2} (\partial_\mu \phi_1)^2 + \frac{1}{2} (\partial_\mu \phi_2)^2 + q \sqrt{\frac{2m^2}{\lambda}} A_\mu \partial^\mu \phi_2 + \frac{q^2}{\lambda} m^2 A^\mu A_\mu + \ldots$$

(21)

The term $\frac{q^2}{\lambda} m^2 A^\mu A_\mu$ resembles a term with a mass governed by the field $A^\mu$ which is the electromagnetic field, so I suppose this term could be interpreted as a massive photon.