COHERENT STATES OF THE HARMONIC OSCILLATOR: OVERLAP OF STATES

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 27.1.

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We'd like the coherent state of the harmonic oscillator to satisfy the classical condition that the position of the oscillator is $\frac{\pi}{2}$ out of phase with its momentum. This condition is satisfied by finding a state $|\alpha\rangle$ that is an eigenstate of the combined operator $Q+iP$ where Q is the position and P is the momentum. These states are known as coherent states, which we've already examined in non-relativistic quantum mechanics. In quantum mechanics, these operators can be written in terms of the creation and annihilation operators

Q and P are actually 'reduced' versions of the usual operators, so that $P=m/\sqrt{2m\omega}$ and $Q=\sqrt{\frac{m\omega}{2}}x$.

$$Q = \frac{1}{\sqrt{2}}\left(a+a^{\dagger}\right) \tag{1}$$

$$P = -\frac{i}{\sqrt{2}}\left(a-a^{\dagger}\right) \tag{2}$$

Thus we're looking for a state $|\alpha\rangle$ that satisfies

$$a|\alpha\rangle = \alpha|\alpha\rangle \tag{3}$$

where α on its own is the eigenvalue and is in general a complex number. This is allowed, since the annihilation operator a is not an observable so need not be hermitian.

L&B show that the state $|\alpha\rangle$ is a superposition of the eigenstates $|n\rangle$ of the harmonic oscillator hamiltonian, where $|n\rangle$ has an energy of $n\omega+\frac{1}{2}$. The state turns out to be L&B's equation 27.2:

$$|\alpha\rangle = e^{-|\alpha|^2/2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle \tag{4}$$

If we have another coherent state $|\beta\rangle$ then it is given by

$$|\beta\rangle = e^{-|\beta|^2/2}\sum_{m=0}^{\infty}\frac{\beta^n}{\sqrt{m!}}|m\rangle \tag{5}$$

The overlap between $|\beta\rangle$ and $|\alpha\rangle$ is then

$$\langle\alpha|\beta\rangle = e^{-|\alpha|^2/2-|\beta|^2/2}\sum_{m,n}\frac{\alpha^{*n}}{\sqrt{n!}}\frac{\beta^m}{\sqrt{m!}}\langle n|m\rangle \tag{6}$$

Since the eigenstates of the harmonic oscillator are orthonormal, we have

$$\langle n|m\rangle = \delta_{nm} \tag{7}$$

so the sum reduces to

$$\langle\alpha|\beta\rangle = e^{-|\alpha|^2/2-|\beta|^2/2}\sum_{m,n}\frac{\alpha^{*n}}{\sqrt{n!}}\frac{\beta^m}{\sqrt{m!}}\delta_{nm} \tag{8}$$

$$= e^{-|\alpha|^2/2-|\beta|^2/2}\sum_{n}\frac{\alpha^{*n}\beta^n}{n!} \tag{9}$$

$$= e^{\alpha^*\beta-|\alpha|^2/2-|\beta|^2/2} \tag{10}$$

Note that if $\alpha=\beta$, then this formula reduces to $\langle\alpha|\alpha\rangle = 1$, so the state is properly normalized.

The difference between the overlap between two different coherent states and two different eigenstates of the hamiltonian arises due to the fact that the coherent state contains contributions from all the eigenstates of the hamiltonian, so any two coherent states will have some non-zero overlap.

Using the relation for the annihilation operator

$$a|n\rangle = \sqrt{n}|n-1\rangle \tag{11}$$

we have, for an eigenstate $|n\rangle$ of the hamiltonian

$$\langle n|a|n\rangle = \sqrt{n}\langle n|n-1\rangle = 0 \tag{12}$$

since the states $|n\rangle$ and $|n-1\rangle$ are orthogonal.

For a coherent state $|\alpha\rangle$, however, we have

$$\langle\alpha|a|\alpha\rangle = \alpha\langle\alpha|\alpha\rangle = \alpha \tag{13}$$

The annihilaton operator a acts on all the states in the sum 4, and converts all the states $|n\rangle$ into states $|n-1\rangle$ with one less energy quantum. However, since this revised sum still contains contributions from all the eigenstates, there is still an overlap with the original state, so the matrix element $\langle\alpha|a|\alpha\rangle \neq 0$.

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