COHERENT STATES OF THE HARMONIC OSCILLATOR:
UNCERTAINTY RELATIONS

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Reference: Tom Lancaster and Stephen J. Blundell, Quantum Field Theory for the Gifted Amateur, (Oxford University Press, 2014), Problems 27.2 - 27.3.
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The coherent states of the harmonic oscillator are eigenvalues of the annihilation operator, so that

\[ a |\alpha\rangle = \alpha |\alpha\rangle \]  

(1)

The reduced position and momentum operators introduced by L&B are:

\[ Q = \frac{1}{\sqrt{2}} (a + a^\dagger) \]  

(2)

\[ P = -i \frac{1}{\sqrt{2}} (a - a^\dagger) \]  

(3)

The expectation values of \( Q \) in a coherent state is then

\[ \langle Q \rangle = \langle \alpha | Q |\alpha\rangle \]  

(4)

\[ = \frac{1}{\sqrt{2}} \left( \langle \alpha | a |\alpha\rangle + \langle \alpha | a^\dagger |\alpha\rangle \right) \]  

(5)

\[ = \frac{1}{\sqrt{2}} (\alpha + \alpha^*) \]  

(6)

\[ = \frac{1}{\sqrt{2}} 2\Re \alpha \]  

(7)

\[ = \sqrt{2} \Re \alpha \]  

(8)

Similarly, for \( P \) we have
\[\langle P \rangle = -\frac{i}{\sqrt{2}} (\alpha - \alpha^*)\]  
\[(9)\]
\[= -\frac{i}{\sqrt{2}} (2i \Im \alpha)\]  
\[(10)\]
\[= \sqrt{2} \Im \alpha\]  
\[(11)\]

Combining these two results, we have

\[\alpha = \frac{1}{\sqrt{2}} (\langle Q \rangle + i \langle P \rangle)\]  
\[(12)\]

From \[2\], we have, using the commutation relation

\[\left[ a, a^\dagger \right] = 1\]  
\[(13)\]

the following results:

\[Q^2 = \frac{1}{2} \left( aa + a^\dagger a^\dagger + aa^\dagger + a^\dagger a \right)\]  
\[(14)\]
\[= \frac{1}{2} \left( aa + a^\dagger a^\dagger + 2a^\dagger a + 1 \right)\]  
\[(15)\]
\[\langle Q^2 \rangle = \langle \alpha | Q^2 | \alpha \rangle\]  
\[(16)\]
\[= \frac{1}{2} \left( \alpha^2 + \alpha^* \right)\]  
\[(17)\]
\[= \frac{1}{2} \left( (\alpha + \alpha^*)^2 + 1 \right)\]  
\[(18)\]
\[= \frac{1}{2} + 2 (\Re \alpha)^2\]  
\[(19)\]

Similarly, for \(P^2\):
\[ P^2 = -\frac{1}{2} \left( a a^\dagger a^\dagger - a a^\dagger - a^\dagger a \right) \]  
\[ = -\frac{1}{2} \left( a a^\dagger a^\dagger - 2 a^\dagger a - 1 \right) \]  
\[ \langle P^2 \rangle = \langle \alpha | P^2 | \alpha \rangle \]  
\[ = -\frac{1}{2} \left( \alpha^2 + \alpha^* 2 - 2 \alpha \alpha^* - 1 \right) \]  
\[ = -\frac{1}{2} \left( (\alpha - \alpha^*)^2 - 1 \right) \]  
\[ = \frac{1}{2} + 2 (\Im \alpha)^2 \]  

The standard deviations are then

\[ \Delta Q = \sqrt{\langle Q^2 \rangle - \langle Q \rangle^2} \]  
\[ = \sqrt{\frac{1}{2} + 2 (\Re \alpha)^2 - \left( \sqrt{2} \Re \alpha \right)^2} \]  
\[ = \frac{1}{\sqrt{2}} \]  

\[ \Delta P = \sqrt{\langle P^2 \rangle - \langle P \rangle^2} \]  
\[ = \sqrt{\frac{1}{2} + 2 (\Im \alpha)^2 - \left( \sqrt{2} \Im \alpha \right)^2} \]  
\[ = \frac{1}{\sqrt{2}} \]  

The uncertainty relation is therefore

\[ (\Delta P)^2 (\Delta Q)^2 = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \]  

which is the minimum uncertainty allowed by the uncertainty principle.

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