COHERENT STATES OF THE HARMONIC OSCILLATOR: UNCERTAINTY RELATIONS

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problems 27.2 - 27.3.

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The coherent states of the harmonic oscillator are eigenvalues of the annihilation operator, so that

$$a|\alpha\rangle = \alpha|\alpha\rangle \tag{1}$$

The reduced position and momentum operators introduced by L&B are:

$$Q = \frac{1}{\sqrt{2}}\left(a + a^{\dagger}\right) \tag{2}$$

$$P = -\frac{i}{\sqrt{2}}\left(a - a^{\dagger}\right) \tag{3}$$

The expectation values of Q in a coherent state is then

The calculations here are similar to those we did earlier for the original x and p operators.

$$\langle Q\rangle = \langle\alpha|Q|\alpha\rangle \tag{4}$$

$$= \frac{1}{\sqrt{2}}\left(\langle\alpha|a|\alpha\rangle + \left\langle\alpha\left|a^{\dagger}\right|\alpha\right\rangle\right) \tag{5}$$

$$= \frac{1}{\sqrt{2}}\left(\alpha + \alpha^{*}\right) \tag{6}$$

$$= \frac{1}{\sqrt{2}}2\Re\alpha \tag{7}$$

$$= \sqrt{2}\Re\alpha \tag{8}$$

Similarly, for P we have

$$\langle P\rangle = -\frac{i}{\sqrt{2}}(\alpha-\alpha^*) \tag{9}$$
$$= -\frac{i}{\sqrt{2}}(2i\Im\alpha) \tag{10}$$
$$= \sqrt{2}\Im\alpha \tag{11}$$

Combining these two results, we have

$$\alpha = \frac{1}{\sqrt{2}}(\langle Q\rangle + i\langle P\rangle) \tag{12}$$

From 2, we have, using the commutation relation

$$\left[a,a^\dagger\right] = 1 \tag{13}$$

the following results:

$$Q^2 = \frac{1}{2}\left(aa+a^\dagger a^\dagger + aa^\dagger + a^\dagger a\right) \tag{14}$$
$$= \frac{1}{2}\left(aa+a^\dagger a^\dagger + 2a^\dagger a + 1\right) \tag{15}$$
$$\left\langle Q^2\right\rangle = \left\langle \alpha\left|Q^2\right|\alpha\right\rangle \tag{16}$$
$$= \frac{1}{2}\left(\alpha^2+\alpha^{*2}+2\alpha\alpha^*+1\right) \tag{17}$$
$$= \frac{1}{2}\left((\alpha+\alpha^*)^2+1\right) \tag{18}$$
$$= \frac{1}{2}+2(\Re\alpha)^2 \tag{19}$$

Similarly, for P^2:

$$P^2 = -\frac{1}{2}\left(aa + a^\dagger a^\dagger - aa^\dagger - a^\dagger a\right) \tag{20}$$
$$= -\frac{1}{2}\left(aa + a^\dagger a^\dagger - 2a^\dagger a - 1\right) \tag{21}$$
$$\left\langle P^2\right\rangle = \left\langle \alpha \left| P^2 \right| \alpha \right\rangle \tag{22}$$
$$= -\frac{1}{2}\left(\alpha^2 + \alpha^{*2} - 2\alpha\alpha^* - 1\right) \tag{23}$$
$$= -\frac{1}{2}\left(\left(\alpha - \alpha^*\right)^2 - 1\right) \tag{24}$$
$$= \frac{1}{2} + 2\left(\Im\alpha\right)^2 \tag{25}$$

The standard deviations are then

$$\Delta Q = \sqrt{\langle Q^2\rangle - \langle Q\rangle^2} \tag{26}$$
$$= \sqrt{\frac{1}{2} + 2\left(\Re\alpha\right)^2 - \left(\sqrt{2}\Re\alpha\right)^2} \tag{27}$$
$$= \frac{1}{\sqrt{2}} \tag{28}$$
$$\Delta P = \sqrt{\langle P^2\rangle - \langle P\rangle^2} \tag{29}$$
$$= \sqrt{\frac{1}{2} + 2\left(\Im\alpha\right)^2 - \left(\sqrt{2}\Im\alpha\right)^2} \tag{30}$$
$$= \frac{1}{\sqrt{2}} \tag{31}$$

The uncertaintly relation is therefore

$$\left(\Delta P\right)^2\left(\Delta Q\right)^2 = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} \tag{32}$$

which is the minimum uncertainty allowed by the uncertainty principle.

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