COHERENT STATES: POSITION AND MOMENTUM WAVE FUNCTIONS

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 27.4.

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A coherent state of the harmonic oscillator is a state $|\alpha\rangle$ that is an eigenstate of the operator

$$\hat{\alpha}=\frac{1}{\sqrt{2}}\left(\hat{Q}+i\hat{P}\right) \tag{1}$$

where $\hat{Q}$ is the position operator and $\hat{P}$ is the momentum operator. In L&B's example 27.3, the derive an expression for the position state wave function $\langle Q|\alpha\rangle$ of the coherent state. The derivation relies on the usual representation of the momentum operator in position space, which is

$$\hat{P}=-i\frac{\partial}{\partial Q} \tag{2}$$

Using this we can write

$$\hat{\alpha}|\alpha\rangle=\alpha|\alpha\rangle=\frac{1}{\sqrt{2}}\left(\hat{Q}+i\hat{P}\right)|\alpha\rangle \tag{3}$$

Operating with $\langle Q|$ on the left, we get

$$\alpha\langle Q|\alpha\rangle=\frac{1}{\sqrt{2}}\left(Q+\frac{\partial}{\partial Q}\right)\langle Q|\alpha\rangle \tag{4}$$

which can be rearranged to give the differential equation

$$\frac{\partial}{\partial Q}\langle Q|\alpha\rangle=-\left(Q-\sqrt{2}\alpha\right)\langle Q|\alpha\rangle \tag{5}$$

The solution of this can be seen (by direct substitution) to be

$$\langle Q|\alpha\rangle=Ce^{-\left(Q-\sqrt{2}\alpha\right)^2/2} \tag{6}$$

where C is a constant determined by normalization. L&B state that $C=1/\pi^{1/4}$, but I don't think this is correct in the general case. To apply normalization we have

$$\langle\alpha|\alpha\rangle = 1 = \int dQ\ \langle\alpha|Q\rangle\langle Q|\alpha\rangle \tag{7}$$

where we've inserted the unit operator in the form

$$1 = \int |Q\rangle\langle Q| \tag{8}$$

From 6, we have (remembering that α is in general complex):

$$\langle\alpha|\alpha\rangle = |C|^2\int dQ\ e^{-\left(Q-\sqrt{2}\alpha\right)^2/2}e^{-\left(Q-\sqrt{2}\alpha^*\right)^2/2} = 1 \tag{9}$$

We can work out this Gaussian integral by multiplying out the exponents and using the usual formula given earlier, or we can just use Maple to do the integral. The result is

$$\langle\alpha|\alpha\rangle = |C|^2\sqrt{\pi}e^{|\alpha|^2-\left(\alpha^2-\alpha^{*2}\right)/2} \tag{10}$$

If α is real, the exponential comes out to just 1 (since the exponent is zero in this case), so we can get a properly normalized state by setting

$$C = \frac{1}{\pi^{1/4}} \tag{11}$$

as is done in L&B's equation 27.13. However, if α is a general complex number, the exponential is *not* 1, and in fact comes out to

$$e^{|\alpha|^2-\left(\alpha^2-\alpha^{*2}\right)/2} = e^{2(\Im\alpha)^2} \tag{12}$$

where $\Im\alpha$ denotes the imaginary part of α. (Note that the exponential is still a real number, though.) Thus the normalization constant should be

$$C = \frac{1}{\pi^{1/4}e^{(\Im\alpha)^2}} \tag{13}$$

To find the momentum space wave function, we use the form of the position operator in momentum space:

$$Q = i\frac{\partial}{\partial P} \tag{14}$$

Using this and following similar steps to the above, we obtain the differential equation

$$\alpha\langle P|\alpha\rangle = \frac{1}{\sqrt{2}}i\left(\frac{\partial}{\partial P}+P\right)\langle P|\alpha\rangle \tag{15}$$

$$\frac{\partial}{\partial P}\langle P|\alpha\rangle = \left(-\sqrt{2}i\alpha - P\right)\langle P|\alpha\rangle \tag{16}$$

This has the general solution

$$\langle P|\alpha\rangle = De^{-\left(P+i\sqrt{2}\alpha\right)^2/2} \tag{17}$$

with D the normalization constant.

Note that the result stated in L&B's exercise 27.4 is missing the division by 2 in the exponent, which is required to be consistent with their result in Equation 27.14.

Doing the normalization integral, we have

$$\langle\alpha|\alpha\rangle = 1 = \int dP\ \langle\alpha|P\rangle\langle P|\alpha\rangle \tag{18}$$

$$= |D|^2\int dP\ e^{-\left(P+i\sqrt{2}\alpha\right)^2/2}e^{-\left(P-i\sqrt{2}\alpha^*\right)^2/2} \tag{19}$$

Again using Maple to do the integral, we get

$$\langle\alpha|\alpha\rangle = \sqrt{\pi}|D|^2 e^{|\alpha|^2+\left(\alpha^2+\alpha^{*2}\right)/2} \tag{20}$$

$$= \sqrt{\pi}|D|^2 e^{2(\Re\alpha)^2} \tag{21}$$

Thus we obtain

$$D = \frac{1}{\pi^{1/4}e^{(\Re\alpha)^2}} \tag{22}$$

In this case, α must be purely imaginary if we are to get L&B's answer of $D = 1/\pi^{1/4}$.