FORCED HARMONIC OSCILLATOR: COHERENT STATE

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Reference: Tom Lancaster and Stephen J. Blundell, *Quantum Field Theory for the Gifted Amateur*, (Oxford University Press, 2014), Problem 27.7.
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We consider again the forced harmonic oscillator with Lagrangian

$$L=\frac{1}{2}m\dot{x}^{2}\left(t\right)-\frac{1}{2}m\omega^{2}x^{2}\left(t\right)+f\left(t\right)x\left(t\right) \tag{1}$$

where $f\left(t\right)$ is a forcing function which acts over the time interval $0<t<T$. The position $\hat{x}\left(t\right)$ is given by the sum of the position $\hat{x}_{0}\left(t\right)$ of the unforced oscillator plus the influence of the forcing function, which is given by

$$\hat{x}\left(t\right)=\hat{x}_{0}\left(t\right)+\int_{0}^{t}dt^{\prime}\chi\left(t-t^{\prime}\right)f\left(t^{\prime}\right) \tag{2}$$

where the response function is given by

$$\chi\left(t-t^{\prime}\right)=\theta\left(t-t^{\prime}\right)\frac{\sin\omega\left(t-t^{\prime}\right)}{m\omega} \tag{3}$$

This can be rewritten as

$$\chi\left(t-t^{\prime}\right)=\theta\left(t-t^{\prime}\right)\frac{e^{i\omega\left(t-t^{\prime}\right)}-e^{-i\omega\left(t-t^{\prime}\right)}}{2im\omega} \tag{4}$$

$$=\theta\left(t-t^{\prime}\right)\frac{i}{2m\omega}\left(e^{-i\omega\left(t-t^{\prime}\right)}-e^{i\omega\left(t-t^{\prime}\right)}\right) \tag{5}$$

We can therefore rewrite 2 using the expression for the position operator in terms of the creation and annihilation operators

$$x=\frac{1}{\sqrt{2m\omega}}\left(\hat{a}+\hat{a}^{\dagger}\right) \tag{6}$$

We assume that the creation operator evolves according to $a^{\dagger}\left(t\right)=e^{i\omega t}a^{\dagger}$ so we get

$$x\left(t\right)=\frac{1}{\sqrt{2m\omega}}\left(\hat{a}e^{-i\omega t}+\hat{a}^{\dagger}e^{i\omega t}\right) \tag{7}$$

Combining this with 5 we get

$$\hat{x}(t) = \frac{1}{\sqrt{2m\omega}}\left(\hat{a}e^{-i\omega t}+\hat{a}^{\dagger}e^{i\omega t}\right)+ \int_0^t dt'\theta\left(t-t'\right)\frac{i}{2m\omega}\left(e^{-i\omega(t-t')}-e^{i\omega(t-t')}\right)f\left(t'\right) \quad (8)$$

We can now define the Fourier transform of the forcing function as

$$\tilde{f}(\omega) = \int_0^T dt' f\left(t'\right)e^{i\omega t'} \quad (9)$$

$$\tilde{f}(-\omega) = \int_0^T dt' f\left(t'\right)e^{-i\omega t'} = \tilde{f}^*(\omega) \quad (10)$$

We then get this expression for $\hat{x}(t)$, provided that $t > T$ so that the full effect of the forcing function has been felt:

$$\hat{x}(t) = \frac{1}{\sqrt{2m\omega}}\left(\hat{a}e^{-i\omega t}+\hat{a}^{\dagger}e^{i\omega t}\right)+ \frac{i}{2m\omega}\left(\tilde{f}(\omega)e^{-i\omega t}-\tilde{f}(-\omega)e^{i\omega t}\right) \quad (11)$$

$$= \frac{1}{\sqrt{2m\omega}}\left[\left(\hat{a}+\frac{i}{\sqrt{2m\omega}}\tilde{f}(\omega)\right)e^{-i\omega t}+ \right. \quad (12)$$

$$\left.\left(\hat{a}^{\dagger}-\frac{i}{\sqrt{2m\omega}}\tilde{f}(-\omega)\right)e^{i\omega t}\right] \quad (13)$$

I think the point of this expression is that if we compare it with 7, we see that the creation and annihilation operators have effectively been replaced by

$$\hat{a} \to \hat{a}+\frac{i}{\sqrt{2m\omega}}\tilde{f}(\omega) \quad (14)$$

$$\hat{a}^{\dagger} \to \hat{a}^{\dagger}-\frac{i}{\sqrt{2m\omega}}\tilde{f}(-\omega) \quad (15)$$

The unforced harmonic oscillator Hamiltonian is

$$\hat{H}_0 = \omega\hat{a}^{\dagger}\hat{a} \quad (16)$$

Technically, there should be a $+\frac{1}{2}$ added on here, but I guess we're supposed to ignore that.

so we can argue that the Hamiltonian for the forced system is the same, but with the modified operators, so we have

$$\hat{H} = \omega\left(\hat{a}^{\dagger}-\frac{i}{\sqrt{2m\omega}}\tilde{f}(-\omega)\right)\left(\hat{a}+\frac{i}{\sqrt{2m\omega}}\tilde{f}(\omega)\right) \quad (17)$$

We can now argue that the ground state for the forced system should be annihilated by the new annihilation operator. If we call this ground state $|\alpha\rangle$ then we have

$$\left(\hat{a}+\frac{i}{\sqrt{2m\omega}}\tilde{f}(\omega)\right)|\alpha\rangle=0 \tag{18}$$

from which we get

$$\hat{a}|\alpha\rangle=-\frac{i}{\sqrt{2m\omega}}\tilde{f}(\omega)|\alpha\rangle \tag{19}$$

Thus the ground state $|\alpha\rangle$ is an eigenstate of the ordinary annihilation operator $\hat{a}$, and thus qualifies as a coherent state, with a value for the eigenvalue of

$$\alpha=-\frac{i}{\sqrt{2m\omega}}\tilde{f}(\omega) \tag{20}$$

The modified number operator is then

$$\hat{n}=\left(\hat{a}^{\dagger}-\frac{i}{\sqrt{2m\omega}}\tilde{f}(-\omega)\right)\left(\hat{a}+\frac{i}{\sqrt{2m\omega}}\tilde{f}(\omega)\right) \tag{21}$$

If we find the expectation value of this new number operator in the state $|n\rangle$ we have

$$\langle n|\hat{n}|n\rangle=\left\langle n\left|\hat{a}^{\dagger}\hat{a}\right|n\right\rangle+\frac{i}{\sqrt{2m\omega}}\tilde{f}(\omega)\left\langle n\left|\hat{a}^{\dagger}\right|n\right\rangle- \\ -\frac{i}{\sqrt{2m\omega}}\tilde{f}(-\omega)\langle n|\hat{a}|n\rangle+\frac{\tilde{f}(-\omega)\tilde{f}(\omega)}{2m\omega}\langle n|n\rangle \tag{22}$$

$$=n+\frac{\left|\tilde{f}(\omega)\right|^2}{2m\omega} \tag{23}$$

The two middle terms in the first equation are zero since the operators $\hat{a}^{\dagger}$ and a both change the number of particles in the state $|n\rangle$ and these states are orthogonal. To get the last term in the last line, we used 10. Thus the number of particles emitted by the forcing term (source) is $\frac{\left|\tilde{f}(\omega)\right|^2}{2m\omega}$. The energy imparted to the system by these particles is just ω times this number, so

$$E'=\frac{\left|\tilde{f}(\omega)\right|^2}{2m} \tag{24}$$

The relation of $|\alpha\rangle$ to the unforced ground state $|0\rangle$ is given by L&B's equation 27.3 which in this case is

$$|\alpha\rangle = e^{-|\alpha|^2/2} e^{\alpha \hat{a}^\dagger} |0\rangle \tag{25}$$

$$= e^{-\left|\tilde{f}(\omega)\right|^2/4m\omega} e^{-i\tilde{f}(\omega)\hat{a}^\dagger/\sqrt{2m\omega}} |0\rangle \tag{26}$$

$$= e^{-E'/2\omega} e^{-i\tilde{f}(\omega)\hat{a}^\dagger/\sqrt{2m\omega}} |0\rangle \tag{27}$$