GEODESIC DEVIATION ON A SPHERE

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Reference: Charles W. Misner, Kip S. Thorne & John Archibald Wheeler, *Gravitation*, W.H. Freeman (1973).

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In Section 1.6, MTW discuss the equation of geodesic deviation by illustrating the deviation between two geodesics (great circles) on a sphere in their Figure 1.10. It might not be immediately obvious where their equation comes from, so I'll try to illustrate here.

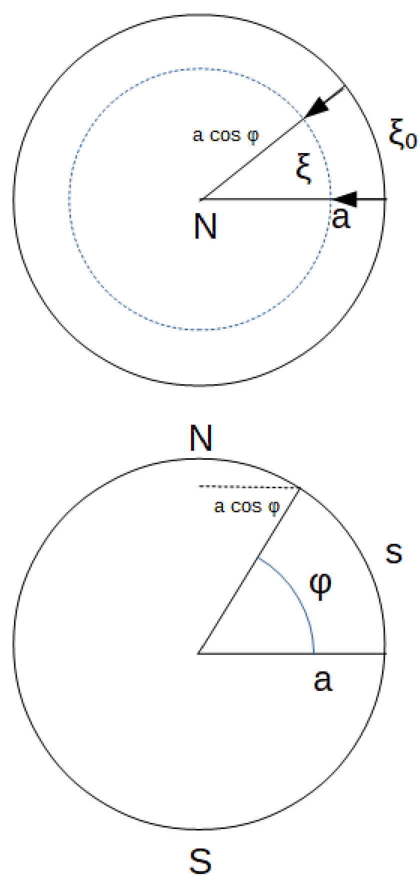

FIGURE 1. Geodesic deviation on a sphere.

See Fig. 1. The top figure is a top-down view of the sphere from above the north pole. The radius of the sphere is a. The horizontal line is the reference (fiducial) line of longitude, which may be thought of as the 0 degree line of longitude on the Earth. The other line making an angle with the first line is another line of longitude. The distance between these two lines measured along the equator of the sphere (the outer circle in the top diagram) is ξ_0. Now suppose we move along both lines in a northerly direction until the new locations both make an angle ϕ with the equator, as shown by the two arrows in the top diagram. In other words, we've now moved up to latitude ϕ. The distance between these two lines of longitude measured along the new line of latitude is ξ, and we wish to find ξ in terms of ξ_0 and ϕ.

Now refer to the bottom diagram. We see that the radius of the horizontal circle at latitude ϕ is $a\cos\phi$. This is also the radius of the dotted circle in the top diagram. By similar figures, we see that the ratio ξ/ξ_0 must be equal to the ratio of the radii of the two circle along which the arc lengths are measured. That is

$$\frac{\xi}{\xi_0} = \frac{a\cos\phi}{a} = \cos\phi \tag{1}$$

Thus we get the desired expression

$$\xi = \xi_0 \cos\phi \tag{2}$$

We can write this in terms of the arc length s we travelled along each line of latitude when we went from the equator to latitude ϕ. As we see in the bottom figure, s is just the arc length when we travel through an angle ϕ in a circle of radius a, so

$$s = a\phi \tag{3}$$

Inserting this into 2, we have

$$\xi = \xi_0 \cos\frac{s}{a} \tag{4}$$

From this we see that ξ, viewed as a function of s, satisfies the simple harmonic motion equation, namely

$$\frac{d^2\xi}{ds^2} + \frac{1}{a^2}\xi = 0 \tag{5}$$

This is a special case of the equation of geodesic deviation on a two-dimensional surface, which is given by MTW's equation 1.6:

$$\frac{d^2\xi}{ds^2} + R\xi = 0 \tag{6}$$

where $R=\frac{1}{a^{2}}$ is the Gaussian curvature of the sphere.

For a more general two-dimensional surface, we have

$$R=\frac{1}{\rho_{1}\rho_{2}} \tag{7}$$

where ρ_{1} and ρ_{2} are the principal radii of curvature of the surface at a specified point. For a sphere, all points are equivalent as far as curvature goes, so $\rho_{1}=\rho_{2}=a$.

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