SPRING TIDE VS NEAP TIDE

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Reference: Charles W. Misner, Kip S. Thorne & John Archibald Wheeler, *Gravitation*, W.H. Freeman (1973). Exercise 1.2.

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The tidal acceleration in Newtonian gravity is given by the equations

$$\frac{d^2\xi}{dt^2} = -\frac{Gm}{r^3}\xi \tag{1}$$

$$\frac{d^2\xi}{dt^2} = \frac{2Gm}{r^3}\xi \tag{2}$$

Eqn 1 gives the acceleration perpendicular to the line connecting the object to the gravitating mass m, and 2 gives the acceleration along the line connecting the object and the mass. Here r is the distance from the object to the centre of the mass m, and ξ is the distance between two observers stationed at a distance r from the mass. These accelerations produce the tidal effects of the mass m. The tidal force perpendicular to the line connecting object and mass 1 tends to pull the two observers towards each other, while that parallel to the connecting line tends to separate them.

We see that these tidal equations are of the same form as the equation of geodesic deviation, which has the general form, for a 2-dimensional surface embedded in Euclidean 3-space

$$\frac{d^2\xi}{ds^2} + R\xi = 0 \tag{3}$$

where R is the Gaussian curvature at each point.

As a taste of things to come later in the book, MTW state that the generalization of 3 to 4-dimensional space time is given by

$$\frac{D^2\boldsymbol{\xi}}{d\tau^2} + \boldsymbol{Riemann}(\boldsymbol{u},\boldsymbol{\xi},\boldsymbol{u}) = 0 \tag{4}$$

where $\boldsymbol{Riemann}$ is the Riemann curvature tensor, τ is the proper time of the object and $\boldsymbol{u}$ is the four-velocity of the object. The capital D in the derivative indicates that we are referring to absolute changes in $\boldsymbol{\xi}$ and not changes that may result from using a particular coordinate system. That is, it's possible for the components of $\boldsymbol{\xi}$ to change in a particular coordinate system, even if $\boldsymbol{\xi}$ itself doesn't change.

This equation can be written in components as

$$\frac{D^2\xi^a}{d\tau^2}+R^{\alpha}_{\beta\gamma\delta}\frac{dx^{\beta}}{d\tau}\xi^{\gamma}\frac{dx^{\delta}}{d\tau}=0 \tag{5}$$

Here $R^{\alpha}_{\beta\gamma\delta}$ represents the components of the Riemann curvature tensor, and $\frac{dx^{\beta}}{d\tau}=u^{\beta}$ is a component of the four-velocity $\boldsymbol{u}$.

In Newtonian physics, the velocities of objects are usually much less than that of light, so the only component of $\boldsymbol{u}$ that is appreciably non-zero is

$$u^0=\frac{dx^0}{d\tau}=1 \tag{6}$$

As a result, the equation of geodesic deviation in Newtonian gravity reduces to MTW's eqn 1.13:

$$\frac{d^2\xi^k}{d\tau^2}+R^k{}_{0j0}\xi^j=0 \tag{7}$$

Comparing this with eqns 1 and 2 shows that the spatial components are

$$\begin{aligned} R^x{}_{0x0}=R^y{}_{0y0}&=\frac{Gm}{r^3}\\ R^z{}_{0z0}&=-\frac{2Gm}{r^3} \end{aligned} \tag{8}$$

with all other spatial components being zero.

These equations are for 'conventional' units, that is, where G and c (the speed of light) are written in conventional units such as grams and centimetres. In the geometrized units used in MTW, we set $G=c=1$ and divide these quantities by c^2, so we have

$$\begin{aligned} R^x{}_{0x0}=R^y{}_{0y0}&=\frac{Gm}{c^2r^3}=\frac{m}{r^3}\\ R^z{}_{0z0}&=-\frac{2Gm}{r^3}=-\frac{2Gm}{c^2r^3}=-\frac{2m}{r^3} \end{aligned} \tag{9}$$

We can work out the units of the components of $R^k{}_{0j0}$ in geometrized units by dimensional analysis. The units of G are $\mathrm{cm^3g^{-1}s^{-2}}$ and of c are $\mathrm{cm}\cdot\mathrm{s^{-1}}$, so we have

$$\text{units of } R^k{}_{0j0}=\frac{\mathrm{cm}^3}{\mathrm{g}\cdot\mathrm{s}^2}\times\frac{\mathrm{s}^2}{\mathrm{cm}^2}\times\frac{\mathrm{g}}{\mathrm{cm}^3}=\mathrm{cm}^{-2} \tag{10}$$

MTW's Box 1.8 shows how quantities can be converted into various units. A mass, for example, can be written conventionally in grams, but then converted to cm by multiplying by $\frac{G}{c^2}$, or to ergs by multiplying by c^2.

We can now work out the tidal effects of the Sun and Moon on the Earth. We need some numbers:

$$G = 6.67 \times 10^{-8}\text{cm}^3\text{g}^{-1}\text{s}^{-2} \tag{11}$$

$$c = 3 \times 10^8\text{cm} \cdot \text{s}^{-1} \tag{12}$$

$$m_☾ = 7.35 \times 10^{25}\text{g} \tag{13}$$

$$r_☾ = 3.84 \times 10^{10}\text{cm} \tag{14}$$

$$m_\odot = 1.989 \times 10^{33}\text{g} \tag{15}$$

$$r_\odot = 1.496 \times 10^{13}\text{cm} \tag{16}$$

First, in conventional units, we have

$$\frac{Gm_☾}{r_☾^3} = 8.658 \times 10^{-14}\text{s}^{-2} \tag{17}$$

$$\frac{Gm_\odot}{r_\odot^3} = 3.96 \times 10^{-14}\text{s}^{-2} \tag{18}$$

From 8, the components of ${R^m}_{0n0}$ are, for the moon

$$\begin{aligned} {R^x}_{0x0☾} &= {R^y}_{0y0☾} = 8.658 \times 10^{-14}\text{s}^{-2} \\ {R^z}_{0z0☾} &= -17.316 \times 10^{-14}\text{s}^{-2} \end{aligned} \tag{19}$$

And for the sun:

$$\begin{aligned} {R^x}_{0x0\odot} &= {R^y}_{0y0\odot} = 3.96 \times 10^{-14}\text{s}^{-2} \\ {R^z}_{0z0\odot} &= -7.92 \times 10^{-14}\text{s}^{-2} \end{aligned} \tag{20}$$

A spring tide occurs when the Earth, Moon and Sun all lie along the same line. In this case, the tidal effects of the Sun and Moon are both due to ${R^z}_{0z0}$ and the effects add. A neap tide occurs when the Earth-Moon line is perpendicular to the Earth-Sun line. Since the strongest tidal effect is due to the Moon, we can use the z component of its tide. The tide due to the Sun is therefore transverse to that of the Moon, so we use its x (or y) component. The ratio of spring to neap tidal accelerations is then

$$\begin{aligned}\frac{\text{spring}}{\text{neap}} &= \frac{R^{z}{}_{0z0☾} + R^{z}{}_{0z0\odot}}{R^{z}{}_{0z0☾} + R^{x}{}_{0x0\odot}} \\ &= \frac{-17.316 - 7.92}{-17.316 + 3.96} \\ &= 1.89\end{aligned} \tag{21}$$

In practice, this is about what is observed in real tide measures.

To convert 19 and 20 to geometrized units, we divide them by c^2 to get, for the Moon:

$$\begin{aligned} R^{x}{}_{0x0☾} &= R^{y}{}_{0y0☾} = 9.62 \times 10^{-35}\text{cm}^{-2} \\ R^{z}{}_{0z0☾} &= -19.24 \times 10^{-35}\text{cm}^{-2}\end{aligned} \tag{22}$$

and for the Sun

$$\begin{aligned} R^{x}{}_{0x0\odot} &= R^{y}{}_{0y0\odot} = 4.403 \times 10^{-35}\text{cm}^{-2} \\ R^{z}{}_{0z0\odot} &= -8.806 \times 10^{-35}\text{cm}^{-2}\end{aligned} \tag{23}$$

Since all terms are multiplied by the same factor, the result 21 is, of course, unchanged.