TENSORS AND ONE-FORMS

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References: Charles W. Misner, Kip S. Thorne & John Archibald Wheeler, *Gravitation*, W.H. Freeman (1973). Exercise 2.1.

Bernard Schutz, *A First Course in General Relativity*, Cambridge U. Press (2009) Section 3.3.

Post date: 6 Jul 2020.

Tensors feature prominently in general relativity, so it's a good idea to get a feel for just what they are. Many books just dive in with the mathematical properties of tensors without giving the reader a feel for what a tensor is. Fortunatley, MTW provides a good introduction to the concept of a tensor.

First, it's important to realize that a tensor, like a vector, is an object that is independent of the coordinate system being used. What *do* depend on the coordinate system are the components of the tensor.

Next, a tensor (written as $\binom{0}{N}$ by Schutz) is actually a linear function of N vectors. The result of this function is to produce an ordinary number after operating on the N vectors. Thus the metric tensor g is actually a function of 2 vectors, with this function providing the scalar product of the two vectors. That is,

$$g\left(\vec{A},\vec{B}\right)=\vec{A}\cdot\vec{B} \tag{1}$$

We're usually used to seeing the scalar product written using the *components* of the metric tensor ina given coordinate system. In a Lorentz frame, for example, we have

$$\vec{A}\cdot\vec{B}=\eta_{\mu\nu}A^{\mu}B^{\nu} \tag{2}$$

The scalar product is independent of the coordinate system (it's the same in every Lorentz frame), and is a function of two vectors, so the metric tensor defined this way is a $\binom{0}{2}$ tensor.

The simplest tensor is a $\binom{0}{0}$ tensor, which is a function of no vectors, and just returns a scalar. Next up the scale we have a $\binom{0}{1}$ tensor, called a one-form or 1-form. According to the definition above, a one-form is a function of a single vector and returns a scalar as its result. MTW begin their discussion of one-forms by considering the geometric and physical pictures of one-forms.

Consider the momentum $\boldsymbol{p}$ of a particle. This is a vector (we're dealing with relativity, so all vectors are 4-vectors). According to quantum mechanics, however, a free particle with momentum $\boldsymbol{p}$ is represented by a plane wave. This wave has evenly spaced peaks and troughs which are described by the phase ϕ of the wave at any point. For a given value of the phase ϕ, we can draw a surface in 4-space of all locations where the phase has that value. The pattern of such surfaces is an example of a one-form.

Now consider two events $\mathscr{P}_0$ and $\mathscr{P}$ and draw the vector between these two points as

$$\boldsymbol{v} = \mathscr{P} - \mathscr{P}_0 \tag{3}$$

This vector will pierce some number of these constant surfaces represented by the one-form. If $\boldsymbol{v}$ connects two points where the phase is equal, then it pierces zero surfaces; if it connects two points with different phases, then the number of surfaces pierced is the difference in phase between these two points.

When we're considering a particle's momentum, the one-form representing the phase is often represented by the symbol $\tilde{\boldsymbol{k}}$. The number of surfaces of $\tilde{\boldsymbol{k}}$ pierced by $\boldsymbol{v}$ is then written

$$\left\langle \tilde{\boldsymbol{k}}, \boldsymbol{v} \right\rangle \tag{4}$$

Schutz uses the notation

$$\tilde{k}(\boldsymbol{v}) \tag{5}$$

for the same thing. Schutz's notation is perhaps clearer, since it emphasizes the fact that a one-form (a tensor, remember) is a function of a single vector.

When dealing with vectors, we usually have to resort to some component representation in a particular coordinate system. Schutz gives us a method for determining the components of a one-form.

In a given coordinate system $\mathscr{O}$, we have a set of basis vectors $\vec{e}_\alpha$. The components of a one-form $\tilde{p}$ are obtained by letting it act on each basis vector. That is

$$p_\alpha \equiv \tilde{p}(\vec{e}_\alpha) \tag{6}$$

From this, we find the expression for the one-form acting on a general vector $\vec{A}$. Since $\vec{A}$ is written in terms of the basis vectors as

$$\vec{A} = A^\alpha \vec{e}_\alpha \tag{7}$$

and the one-form is a linear function, we have, since the components A^α are just numbers:

$$\tilde{p}\left(\vec{A}\right) = \tilde{p}\left(A^{\alpha}\vec{e}_{\alpha}\right) \tag{8}$$
$$= A^{\alpha}\tilde{p}\left(\vec{e}_{\alpha}\right) \tag{9}$$
$$= A^{\alpha}p_{\alpha} \tag{10}$$

We can use this result to obtain a set of basis one-forms in a given frame $\mathcal{O}$. We would like to be able to write a one-form in terms of a set $\tilde{\omega}^{\alpha}$ of basis one-forms as

$$\tilde{p} = p_{\alpha}\tilde{\omega}^{\alpha} \tag{11}$$

As it stands, this is an abstract equation, since remember that a one-form is a function of a vector. Thus in order for 11 to make sense, we have to use it to act on a vector. Let's applying to the vector $\vec{A}$ above. We get

$$\tilde{p}\left(\vec{A}\right) = p_{\alpha}\tilde{\omega}^{\alpha}\left(\vec{A}\right) \tag{12}$$
$$= p_{\alpha}\tilde{\omega}^{\alpha}\left(A^{\beta}\vec{e}_{\beta}\right) \tag{13}$$
$$= p_{\alpha}A^{\beta}\tilde{\omega}^{\alpha}\left(\vec{e}_{\beta}\right) \tag{14}$$

Notice that this formula involves two sums; one over α and one over β. Also note that we have a set of 4 $\tilde{\omega}^{\alpha}$s, each of which acts on each of the 4 basis vectors $\vec{e}_{\beta}$. Each one-form has 4 components, as given by 6. Now in order for 14 to be consistent with 10 for any vector $\vec{A}$, we must have

$$\tilde{\omega}^{\alpha}\left(\vec{e}_{\beta}\right) = \delta^{\alpha}{}_{\beta} \tag{15}$$

That is, the components of $\tilde{\omega}^{\alpha}$ in the coordinate system $\mathcal{O}$ must be

$$\begin{aligned} \tilde{\omega}^{0} &\rightarrow (1,0,0,0) \\ \tilde{\omega}^{1} &\rightarrow (0,1,0,0) \\ \tilde{\omega}^{2} &\rightarrow (0,0,1,0) \\ \tilde{\omega}^{3} &\rightarrow (0,0,0,1) \end{aligned} \tag{16}$$

These are the basis one-forms relative to the basis vectors $\vec{e}_{\alpha}$. If we used different basis vectors, we'd get different basis one-forms.

Finally, we revert back to MTW to see the relation between a vector $\boldsymbol{p}$ and a corresponding one-form $\tilde{\boldsymbol{p}}$. In quantum mechanics, the phase becomes the momentum when it is multiplied by $\hbar$ (in relativistic quantum theory, we usually take $\hbar = 1$, so then momentum and phase become the same). If we multiply $\tilde{\boldsymbol{k}}$ above by $\hbar$, we then get the momentum one-form $\tilde{\boldsymbol{p}}$. If we now use this one-form to act on some vector $\boldsymbol{v}$, we get

$$\langle \tilde{\boldsymbol{p}}, \boldsymbol{v} \rangle \tag{17}$$

which is the number of surfaces of $\tilde{\boldsymbol{p}}$ pierced by $\boldsymbol{v}$. This is just another way of saying that we are finding the projection of $\boldsymbol{v}$ onto the vector $\boldsymbol{p}$, which is given by the scalar product. In other words,

$$\boldsymbol{p} \cdot \boldsymbol{v} = \langle \tilde{\boldsymbol{p}}, \boldsymbol{v} \rangle \tag{18}$$

In quantum mechanics, the quantity $e^{ip\cdot x}$ is often found. This can be expanded to

$$e^{ip\cdot x} = \exp\left[i\left(\boldsymbol{k}\cdot\boldsymbol{x} - \omega t\right)\right] \tag{19}$$

The exponent (apart from the i) is the phase difference $\Delta\phi$ between two events $\mathscr{X}_0$ and $\mathscr{X}$ so we see that in this case

$$p \cdot x = \langle \tilde{\boldsymbol{p}}, x \rangle = \Delta\phi \tag{20}$$

That is, the projection of the 4-vector p onto the 4-vector x separating the two events $\mathscr{X}_0$ and $\mathscr{X}$ (which is $p \cdot x$) is equivalent to the number of phase surfaces pierced by the vector x. This isn't a rigorous proof, of course, but it illustrates conceptually how we can consider a one-form and its corresponding vector to be equivalent.

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