COMPONENTS OF ONE-FORMS AND VECTORS

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References: Charles W. Misner, Kip S. Thorne & John Archibald Wheeler, *Gravitation*, W.H. Freeman (1973). Exercises 2.2 - 2.4.

Bernard Schutz, *A First Course in General Relativity*, Cambridge U. Press (2009) Section 3.3.

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A one-form can be written in terms of its components in a particular frame of reference. If a frame has as its basis vectors $\boldsymbol{e}_\alpha$, then the basis one-forms satisfy

$$\boldsymbol{\omega}^\alpha\left(\boldsymbol{e}_\beta\right)=\left\langle\boldsymbol{\omega}^\alpha,\boldsymbol{e}_\beta\right\rangle=\delta_\beta^\alpha \tag{1}$$

I've reverted to using MTW's notation for basis vectors and one-forms.

A one-form can then be written in terms of its components and basis one-forms as

$$\boldsymbol{\sigma}=\sigma_\alpha\boldsymbol{\omega}^\alpha \tag{2}$$

Remembering that the expression

$$\langle\boldsymbol{\sigma},\boldsymbol{v}\rangle \tag{3}$$

is interpreted as the number of surfaces of the one-form $\boldsymbol{\sigma}$ that are pierced by the vector $\boldsymbol{v}$, we can find how many surfaces of some one-form $\boldsymbol{\sigma}$ are pierced by the basis vector $\boldsymbol{e}_\alpha$:

$$\begin{aligned}\left\langle\boldsymbol{\sigma},\boldsymbol{e}_\beta\right\rangle &= \left\langle\sigma_\alpha\boldsymbol{\omega}^\alpha,\boldsymbol{e}_\beta\right\rangle &(4)\\ &= \sigma_\alpha\left\langle\boldsymbol{\omega}^\alpha,\boldsymbol{e}_\beta\right\rangle &(5)\\ &= \sigma_\alpha\delta_\beta^\alpha &(6)\\ &= \sigma_\beta &(7)\end{aligned}$$

The second line follows from the linearity of one-forms and the third line from 1.

We can do a similar calculation to find the the number of surfaces of the basis one-form $\boldsymbol{\omega}^\alpha$ that are pierced by an arbitrary vector $\boldsymbol{v}$.

$$\langle \boldsymbol{\omega}^{\alpha}, \boldsymbol{v} \rangle = \left\langle \boldsymbol{\omega}^{\alpha}, v^{\beta} \boldsymbol{e}_{\beta} \right\rangle \tag{8}$$
$$= v^{\beta} \left\langle \boldsymbol{\omega}^{\alpha}, \boldsymbol{e}_{\beta} \right\rangle \tag{9}$$
$$= v^{\beta} \delta^{\alpha}{}_{\beta} \tag{10}$$
$$= v^{\alpha} \tag{11}$$

Note that the components of one-forms have subscripts to label them, while the components of vectors have superscripts. If we combine 7 and 10 we find

$$\langle \boldsymbol{\sigma}, \boldsymbol{v} \rangle = \left\langle \sigma_{\alpha} \boldsymbol{\omega}^{\alpha}, v^{\beta} \boldsymbol{e}_{\beta} \right\rangle \tag{12}$$
$$= \sigma_{\alpha} v^{\beta} \left\langle \boldsymbol{\omega}^{\alpha}, \boldsymbol{e}_{\beta} \right\rangle \tag{13}$$
$$= \sigma_{\alpha} v^{\beta} \delta^{\alpha}{}_{\beta} \tag{14}$$
$$= \sigma_{\alpha} v^{\alpha} \tag{15}$$

This is just another way of writing the scalar product of the vector corresponding to the one-form $\boldsymbol{\sigma}$ with the vector $\boldsymbol{v}$.

Ex. 2.2. From MTW's eqn 2.11, the scalar product of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is written in terms of the metric $\eta_{\alpha\beta}$ as

$$\boldsymbol{u} \cdot \boldsymbol{v} = u^{\alpha} v^{\beta} \eta_{\alpha\beta} \tag{16}$$

Since the one-form $\tilde{\boldsymbol{u}}$ corresponding to the vector $\boldsymbol{u}$ satisfies 15 (with $\boldsymbol{\sigma}$ replaced by $\boldsymbol{u}$), we have

$$u_{\beta} v^{\beta} = u^{\alpha} v^{\beta} \eta_{\alpha\beta} \tag{17}$$

Since this must be true for all vectors $\boldsymbol{v}$, we must have

$$u_{\beta} = u^{\alpha} \eta_{\alpha\beta} \tag{18}$$

The metric used in MTW is

$$\begin{aligned} \eta_{00} &= -1 \\ \eta_{ii} &= +1 \end{aligned} \tag{19}$$

with all other elements being zero. Therefore we can get the components of a one-form from the components of the corresponding vector:

$$\begin{aligned} u_0 &= -u^0 \\ u_i &= u^i \end{aligned} \tag{20}$$

Ex. 2.3. Since the metric $\eta_{\alpha\beta}$ multiplied by itself gives the unit matrix, $\eta_{\alpha\beta}$ is its own inverse. That is

$$\left\| \eta^{\alpha\beta} \right\| = \left\| \eta_{\alpha\beta} \right\|^{-1} = \left\| \eta_{\alpha\beta} \right\| \tag{21}$$

where the notation $\left\| \eta_{\alpha\beta} \right\|$ means 'the matrix with elements $\eta_{\alpha\beta}$. We can therefore multiply 18 by $\eta^{\beta\gamma}$ to get

$$u_\beta \eta^{\beta\gamma} = u^\alpha \eta_{\alpha\beta} \eta^{\beta\gamma} \tag{22}$$

$$= u^\alpha \delta^\gamma{}_\alpha \tag{23}$$

$$= u^\gamma \tag{24}$$

That is, we can use the inverse metric to raise an index to recover a vector's component from the components of its corresponding one-form.

Ex. 2.4. Finally, we see there are several ways to write the scalar product of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$. One way is given in 15:

$$\boldsymbol{u} \cdot \boldsymbol{v} = \langle \boldsymbol{u}, \boldsymbol{v} \rangle = u_\alpha v^\alpha \tag{25}$$

Using 18 and 24 we can raise or lower the index on one of the vector components to get

$$\boldsymbol{u} \cdot \boldsymbol{v} = u_\alpha v_\beta \eta^{\alpha\beta} = u^\alpha v^\beta \eta_{\alpha\beta} \tag{26}$$

PINGBACKS

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