ENERGY AND VELOCITY FROM 4-MOMENTUM

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References: Charles W. Misner, Kip S. Thorne & John Archibald Wheeler, *Gravitation*, W.H. Freeman (1973). Exercise 2.5.

Bernard Schutz, *A First Course in General Relativity*, Cambridge U. Press (2009) Section 3.3.

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We are given a particle of rest mass m and 4-momentum p and an observer with 4-velocity u.

Ex 2.5a. The energy measured by the observer in his rest frame can be obtained by noting that in this frame, his 4-velocity is

I'm using ordinary non-bold font to represent 4-vectors and one-forms, and bold font for 3-vectors.

$$u=(1,0,0,0) \tag{1}$$

and the energy of the particle is its 0 or time component p^0. Therefore, in the observer's rest frame (remember MTW's metric is $(-1,1,1,1)$):

$$E=p^0=-p\cdot u \tag{2}$$

Since this is a scalar product, it is valid in all inertial frames.

Ex 2.5b. From MTW's Box 2.2 we have

$$u^2=-1 \tag{3}$$

and

$$p=mu \tag{4}$$

so

$$p^2=p\cdot p \tag{5}$$

$$=m^2u\cdot u \tag{6}$$

$$=-m^2 \tag{7}$$

Ex 2.5c. Again, from Box 2.2 we have

$$p^2 = -E^2 + \boldsymbol{p}^2 \tag{8}$$

where $\boldsymbol{p}$ (bold font) is the 3-momentum. Therefore

$$|\boldsymbol{p}| = \sqrt{E^2 + p^2} \tag{9}$$

$$= \sqrt{(p \cdot u)^2 + (p \cdot p)^2} \tag{10}$$

Ex 2.5d. From Box 2.2, we have

$$E^2 = \frac{m^2}{1 - v^2} \tag{11}$$

$$\boldsymbol{p}^2 = \frac{m^2 v^2}{1 - v^2} \tag{12}$$

so

$$\frac{\boldsymbol{p}^2}{E^2} = v^2 = \boldsymbol{v} \cdot \boldsymbol{v} \tag{13}$$

so

$$|\boldsymbol{v}| = \frac{|\boldsymbol{p}|}{E} \tag{14}$$

Ex 2.5e. The 4-vector we're asked to find has components

$$v = (0, \boldsymbol{v}) \tag{15}$$

which is a rather unusual object. The question states that v is to be measured in the observer's Lorentz frame, which presumably means we can take u to be given by 1, that is, the observer's rest frame. In that frame, MTW's eqn 2.35 becomes

$$\frac{p + (p \cdot u)\, u}{-p \cdot u} = \frac{1}{E}\left(\left(E, \frac{m\boldsymbol{v}}{\sqrt{1 - \boldsymbol{v}^2}}\right) - (E, 0, 0, 0)\right) \tag{16}$$

$$= \frac{1}{E}\left(0, \frac{m\boldsymbol{v}}{\sqrt{1 - \boldsymbol{v}^2}}\right) \tag{17}$$

$$= (0, \boldsymbol{v}) \tag{18}$$

where the last line uses

$$E = \frac{m}{\sqrt{1 - \boldsymbol{v}^2}} \tag{19}$$

We can also verify that MTW's eqn 2.35 gives the correct magnitude for the 4-vector v. We have

$$v\cdot v = \frac{p\cdot p+2\left(p\cdot u\right)^{2}+\left(p\cdot u\right)\left(u\cdot u\right)}{\left(p\cdot u\right)^{2}} \tag{20}$$

$$= \frac{-m^{2}+2\left(p\cdot u\right)^{2}-\left(p\cdot u\right)}{\left(p\cdot u\right)^{2}} \tag{21}$$

$$= \frac{\left(p\cdot u\right)^{2}-m^{2}}{\left(p\cdot u\right)^{2}} \tag{22}$$

Starting with the definition of $p=mv$, $E=-p\cdot u$ and $E^{2}=m^{2}+\boldsymbol{p}^{2}$ we have

$$E^{2} = m^{2}+\frac{m^{2}\boldsymbol{v}^{2}}{1-\boldsymbol{v}^{2}} \tag{23}$$

$$= \frac{m^{2}}{1-\boldsymbol{v}^{2}} \tag{24}$$

$$= \left(p\cdot u\right)^{2} \tag{25}$$

Solving for $\boldsymbol{v}^{2}$ we have

$$\boldsymbol{v}^{2} = \frac{\left(p\cdot u\right)^{2}-m^{2}}{\left(p\cdot u\right)^{2}} \tag{26}$$

which agrees with 22, since $v\cdot v=(0,\boldsymbol{v})\cdot(0,\boldsymbol{v})=\boldsymbol{v}^{2}$.

Incidentally, we can also show that

$$u\cdot v = \frac{u\cdot p+\left(p\cdot u\right)u^{2}}{-p\cdot u} \tag{27}$$

$$= \frac{u\cdot p-p\cdot u}{-p\cdot u} \tag{28}$$

$$= 0 \tag{29}$$

Since this is a scalar product, it is valid in all inertial frames. I'm not sure what the significance of this is, but it's interesting.