GRADIENT AS A ONE-FORM

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References: Charles W. Misner, Kip S. Thorne & John Archibald Wheeler, *Gravitation*, W.H. Freeman (1973). Exercise 2.6.

Bernard Schutz, *A First Course in General Relativity*, Cambridge U. Press (2009) Section 3.3.

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A one-form $\tilde{p}$ is a tensor that acts on a single vector A according to

$$\tilde{p}\left(\vec{A}\right)=A^{\alpha}p_{\alpha} \tag{1}$$

where A^{α} is a component of the vector A and p_{α} is a component of the one-form $\tilde{p}$.

Following Schutz, we can see that the gradient of a scalar field ϕ satisfies the definition of a one-form. Suppose that ϕ is defined at every spacetime event along some path. The path is one followed by an observer, so we can consider the situation in the frame of the observer. Within this frame, we can label every event along the path with the observer's proper time τ. In some other frame, the coordinates of a particular event are given by the coordinates t, x, y and z. Each of these coordinates can be written as a function of τ, so we have

$$\begin{aligned}
t &= t(\tau)\\
x &= x(\tau)\\
y &= y(\tau)\\
z &= z(\tau)
\end{aligned} \tag{2}$$

Suppose we now wish to find the rate of change of ϕ along the path of the observer, where by 'rate of change' we mean $d\phi/d\tau$, that is, the rate of change as viewed by the observer himself. We can calculate this in any frame, by using 2. We have

$$\frac{d\phi}{d\tau}=\frac{\partial\phi}{\partial t}\frac{dt}{d\tau}+\frac{\partial\phi}{\partial x}\frac{dx}{d\tau}+\frac{\partial\phi}{\partial y}\frac{dy}{d\tau}+\frac{\partial\phi}{\partial z}\frac{dz}{d\tau} \tag{3}$$

The set of 4 derivatives with respect to τ form the definition of the 4-velocity u:

$$u=\left[\frac{dt}{d\tau},\frac{dx}{d\tau},\frac{dy}{d\tau},\frac{dz}{d\tau}\right] \tag{4}$$

If we write the partial derivatives in 3 as components, we have

$$\tilde{d}\phi=\left[\frac{\partial\phi}{\partial t},\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y},\frac{\partial\phi}{\partial z}\right] \tag{5}$$

By comparing this with 1, we see that 3 can be written as

$$\frac{d\phi}{d\tau}=u^{\alpha}\left(\tilde{d}\phi\right)_{\alpha} \tag{6}$$

The quantity $\tilde{d}\phi$ therefore satisfies the definition of a one-form. From its definition in 5, we see that it is also the usual definition of the gradient (in spacetime). Thus the gradient is a one-form.

In MTW's notation, we would write 3 as

$$\frac{d\phi}{d\tau}=\langle\boldsymbol{d}\phi,u\rangle=\partial_u\phi \tag{7}$$

This is the rate of change of a function ϕ along a vector u.

Ex. 2.6. As an example, we can take the scalar field ϕ to be the temperature T inside the Sun, and the 4-velocity u to be that of a cosmic ray particle as it travels through the Sun. As this fits the definitions given above, the rate of change of the temperature as seen by the cosmic ray is given by 7, with $\phi\to T$:

$$\frac{dT}{d\tau}=\langle\boldsymbol{d}T,u\rangle=\partial_u T \tag{8}$$

In the frame of the cosmic ray, $t=\tau$ and

$$u=[1,0,0,0] \tag{9}$$

so 3 reduces to the identity

$$\frac{dT}{d\tau}=\frac{\partial T}{\partial t}\times 1=\frac{dT}{d\tau} \tag{10}$$

In the frame of the Sun, 3 becomes

$$\frac{dT}{d\tau}=\frac{\partial T}{\partial t}u^0+\frac{\partial T}{\partial x^i}u^i \tag{11}$$

$$=\frac{1}{\sqrt{1-v^2}}\frac{\partial T}{\partial t}+\frac{v^i}{\sqrt{1-v^2}}\frac{\partial T}{\partial x^i} \tag{12}$$

This is reasonable because, as seen from the rest frame of the Sun, the cosmic ray observes temperature changes due to two effects. First, the temperature of a particular location in the Sun could be changing on its own, and second, the temperature could be different in different locations, so the cosmic ray would experience a change due to its changing location.