LORENTZ BOOST IN AN ARBITRARY DIRECTION

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: Charles W. Misner, Kip S. Thorne & John Archibald Wheeler, *Gravitation*, W.H. Freeman (1973). Exercise 2.7.

Post date: 17 Jul 2020.

Usually we see Lorentz transformations for a boost parallel to one of the coordinate axes. MTW give the general form for a boost in an arbitrary direction. In matrix form, from an unprimed system to a primed system, we have

$$\Lambda=\left[\begin{array}{cccc}\gamma & -\beta\gamma n^{1} & -\beta\gamma n^{2} & -\beta\gamma n^{3}\\ -\beta\gamma n^{1} & 1+(\gamma-1)\left(n^{1}\right)^{2} & (\gamma-1)n^{1}n^{2} & (\gamma-1)n^{1}n^{3}\\ -\beta\gamma n^{2} & (\gamma-1)n^{1}n^{2} & 1+(\gamma-1)\left(n^{2}\right)^{2} & (\gamma-1)n^{2}n^{3}\\ -\beta\gamma n^{3} & (\gamma-1)n^{1}n^{3} & (\gamma-1)n^{2}n^{3} & 1+(\gamma-1)\left(n^{3}\right)^{2}\end{array}\right] \tag{1}$$

Here

$$\gamma=\frac{1}{\sqrt{1-\beta^{2}}} \tag{2}$$

with β being the relative speed as usual, and

$$\boldsymbol{n}=\left[n^{1},n^{2},n^{3}\right] \tag{3}$$

is a unit vector, so that

$$\left(n^{1}\right)^{2}+\left(n^{2}\right)^{2}+\left(n^{3}\right)^{2}=1 \tag{4}$$

Ex 2.7(a). We are first asked to verify that Λ satisfies the relation

$$\Lambda^{T}\eta\Lambda=\eta \tag{5}$$

where η is the flat space metric

$$\eta=\left[\begin{array}{cccc}-1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{array}\right] \tag{6}$$

I'm not aware of any fast way of doing this, so it seems we need to resort to brute force. I used Maple to do the matrix multiplications and simplification using 2 and 4. The result is that 5 is, in fact, true.

I'll give a few of the matrix elements of 5 so you can see how they simplify. Defining

$$L \equiv \Lambda^T \eta \Lambda \tag{7}$$

we have

$$\begin{align} L[0,0] &= \beta^2\gamma^2 \left(n^1\right)^2 + \beta^2\gamma^2 \left(n^2\right)^2 + \beta^2\gamma^2 \left(n^3\right)^2 + -\gamma^2 \tag{8}\\ &= \gamma^2 \left(\beta^2 \left(\left(n^1\right)^2 + \left(n^2\right)^2 + \left(n^3\right)^2\right) - 1\right) \tag{9}\\ &= \gamma^2 \left(\beta^2 - 1\right) \tag{10}\\ &= \gamma^2 \left(-\frac{1}{\gamma^2}\right) \tag{11}\\ &= -1 \tag{12} \end{align}$$

Another example:

$$\begin{align} L[1,2] &= -\beta^2\gamma^2 n^1 n^2 + \left(1 + (\gamma-1)\left(n^1\right)^2\right)(\gamma-1)\, n^1 n^2 + \tag{13}\\ &\quad (\gamma-1)\, n^1 n^2 \left(1 + (\gamma-1)\left(n^2\right)^2\right) + (\gamma-1)^2 n^1 \left(n^3\right)^2 n^2 \tag{14}\\ &= n^1 n^2 \left[-\beta^2\gamma^2 + 2(\gamma-1) + (\gamma-1)^2 \left(\left(n^1\right)^2 + \left(n^2\right)^2 + \left(n^3\right)^2\right)\right] \tag{15}\\ &= n^1 n^2 \left[-\beta^2\gamma^2 + 2(\gamma-1) + (\gamma-1)^2\right] \tag{16}\\ &= n^1 n^2 \left[\gamma^2\left(1-\beta^2\right) + 2\gamma - 2\gamma - 2 + 1\right] \tag{17}\\ &= n^1 n^2 [1 - 2 + 1] \tag{18}\\ &= 0 \tag{19} \end{align}$$

The other elements of L can be worked out similarly, if you have the patience. I'd imagine that you would only need to work out $L[0,0]$, one of the other diagonal elements and one of the off-diagonal elements, since the others would be obtained by permuting indices.

Ex 2.7(b&c) . The matrix 1 transforms from the unprimed to the primed system, so we have

$$x^{\alpha'} = \Lambda^{\alpha'}{}_{\beta} x^{\beta} \tag{20}$$

To work out the relative velocity of the two frames, we would like to find $dx^{i'}/dx^{0'}$. Using the entries from 1, we have (repeated Latin indexes are summed over 1 to 3):

$$dx^{i'} = (\gamma-1)\,n^i n^j dx^j + dx^i - \beta\gamma n^i dx^0 \tag{21}$$

$$dx^{0'} = \gamma dx^0 - \beta\gamma n^j dx^j \tag{22}$$

We can now multiply 21 on both sides by n^i and sum over i, using 4:

$$n^i dx^{i'} = (\gamma-1)\,n^i n^i n^j dx^j + n^i dx^i - \beta\gamma n^i n^i dx^0 \tag{23}$$

$$= (\gamma-1)\,n^j dx^j + n^i dx^i - \beta\gamma dx^0 \tag{24}$$

We also multiply 22 on both sides by β:

$$\beta dx^{0'} = \beta\gamma dx^0 - \beta^2\gamma n^j dx^j \tag{25}$$

Adding this to 24 we have

$$n^i dx^{i'} + \beta dx^{0'} = (\gamma-1)\,n^j dx^j + n^i dx^i - \beta^2\gamma n^j dx^j \tag{26}$$

$$= \gamma\left(1-\beta^2\right) n^i dx^i \tag{27}$$

$$= \frac{1}{\gamma} n^i dx^i \tag{28}$$

where we relabelled the dummy index j to i, since it's summed.

Now if we want the relative velocity of the two frames, we want to see how the location of a single point in one frame is viewed in the other. That is, we can take $dx^i = 0$ in the unprimed frame to see how fast a point in the unprimed frame is seen to move in the primed frame. Doing this gives us

$$n^i \frac{dx^{i'}}{dx^{0'}} = -\beta \tag{29}$$

Defining the vector $\boldsymbol{v}'$ to be

$$\boldsymbol{v}' \equiv \left[\frac{dx^{1'}}{dx^{0'}}, \frac{dx^{2'}}{dx^{0'}}, \frac{dx^{3'}}{dx^{0'}}\right] \tag{30}$$

we have

$$\boldsymbol{n}\cdot\boldsymbol{v}' = -\beta \tag{31}$$

Since $\boldsymbol{n}$ is a unit vector, we can take

$$\boldsymbol{v}' = -\beta\left[n^1, n^2, n^3\right] = -\beta\boldsymbol{n} \tag{32}$$

Strictly speaking, $\boldsymbol{v}'$ could also have a component perpendicular to $\boldsymbol{n}$, since this would disappear in the dot product. However, movement perpendicular to the relative velocity is not affected by a Lorentz transformation.

To find the velocity of the primed frame as seen in the unprimed frame, we just replace β by $-\beta$ in the above argument and everything works out the same except for the flipped sign, so we have

$$\boldsymbol{v} = \beta\boldsymbol{n} \tag{33}$$

Ex 2.7(d). To verify that 1 reduces to the usual Lorentz transformation if the velocity is in the z direction, we set $n^1 = n^2 = 0$ and $n^3 = 1$ which gives

$$\left\|\Lambda^{\nu'}{}_{\mu}\right\| = \begin{bmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{bmatrix} \tag{34}$$

This is the transformation from the unprimed to the primed system. To get the inverse transformation, we replace β by $-\beta$:

$$\left\|\Lambda^{\mu}{}_{\nu'}\right\| = \begin{bmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{bmatrix} \tag{35}$$

PINGBACKS

Pingback: Inertial mass per unit volume