LORENTZ FORCE LAW AND THE ELECTROMAGNETIC FIELD TENSOR

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References: Charles W. Misner, Kip S. Thorne & John Archibald Wheeler, *Gravitation*, W.H. Freeman (1973). Exercise 3.1.

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The Lorentz force law is the expression of the classical electromagnetic force on a test charge e due to an electric field $\boldsymbol{E}$ and a magnetic field $\boldsymbol{B}$. If the charge's velocity is $\boldsymbol{v}$, the force, written as the derivative of the charge's momentum $\boldsymbol{p}$ is

$$\frac{d\boldsymbol{p}}{dt}=e\left(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}\right) \tag{1}$$

The problem with this statement of the Lorentz law is that it describes the behaviour of the charge only in the three dimensions of the momentum $\boldsymbol{p}$, rather than the four dimensions of energy-momentum p required in relativity. To provide a description that is independent of the observer's frame of reference, we want the derivative of the four-momentum p with respect to the proper time τ of the test charge, that is, we want an expression of the form $\frac{dp}{d\tau}=\ldots$. Using relativity, the force in the particle's rest frame is related to the force in the observer's frame (in which the velocity of the particle is $\boldsymbol{v}$) is given by

$$\frac{d\boldsymbol{p}}{d\tau}=\frac{1}{\sqrt{1-\boldsymbol{v}^{2}}}\frac{d\boldsymbol{p}}{dt} \tag{2}$$

$$=\frac{1}{\sqrt{1-\boldsymbol{v}^{2}}}e\left(\boldsymbol{E}+\boldsymbol{v}\times\boldsymbol{B}\right) \tag{3}$$

This can be written in terms of the components of the four-velocity u as

$$\frac{d\boldsymbol{p}}{d\tau}=e\left(u^{0}\boldsymbol{E}+\boldsymbol{u}\times\boldsymbol{B}\right) \tag{4}$$

What's missing from this equation is the zeroth component of the four-momentum, that is, p^0, which is the energy. The energy of the test charge (due to electromagnetic forces; we're not looking at its rest mass) is the work done to move the charge to its current location, which is

$$E=\boldsymbol{E}\cdot\boldsymbol{x} \tag{5}$$

Be careful with the notation here. The symbol E (non-bold) is the energy; the bold face $\boldsymbol{E}$ is the electric field vector.

Since the magnetic field always acts perpendicularly to the direction of motion, it does no work on the charge, so all the change in energy is due to the electric field. This is

$$\frac{dp^0}{d\tau} = \frac{1}{\sqrt{1-\boldsymbol{v}^2}}\frac{dE}{dt} \tag{6}$$

$$= \frac{1}{\sqrt{1-\boldsymbol{v}^2}} e\boldsymbol{E}\cdot\frac{d\boldsymbol{x}}{dt} \tag{7}$$

$$= \frac{1}{\sqrt{1-\boldsymbol{v}^2}} e\boldsymbol{E}\cdot\boldsymbol{v} \tag{8}$$

$$= e\boldsymbol{E}\cdot\boldsymbol{u} \tag{9}$$

We wish to combine 4 and 9 into a single equation. First, we note that these equations are linear in u. This implies that we can define what MTW call a 'linear machine' which takes in the four-velocity u and outputs a 4-vector $\frac{dp}{d\tau}$. They name this machine $\boldsymbol{Faraday}$, although it is more commonly known as the *electromagnetic field tensor*. That is, we have

$$\frac{dp}{d\tau} = eF(u) \tag{10}$$

where $F(u)$ is the EM field tensor. They define the components of F by the condition

$$\frac{dp^\alpha}{d\tau} = eF^\alpha{}_\beta u^\beta \tag{11}$$

Ex 3.1. We can find these components in terms of the fields $\boldsymbol{E}$ and $\boldsymbol{B}$ by comparing 11 with 4 and 9. Consider first $dp^0/d\tau$. We have from 9

I use 1, 2, 3 as indexes instead of MTW's x, y, z.

$$\frac{dp^0}{d\tau} = e\left(E_1u^1 + E_2u^2 + E_3u^3\right) \tag{12}$$

Comparing this with 11 we see that

$$\begin{aligned} F^0{}_0 &= 0 \\ F^0{}_1 &= E_1 \\ F^0{}_2 &= E_2 \\ F^0{}_3 &= E_3 \end{aligned} \tag{13}$$

Now consider, using 4

$$\begin{aligned}\frac{dp^1}{d\tau} &= e\left(u^0E_1 + (\boldsymbol{u}\times\boldsymbol{B})_1\right)\\ &= e\left(u^0E_1 + u^2B_3 - u^3B_2\right)\end{aligned} \tag{14}$$

From this we get

$$\begin{aligned}F^1{}_0 &= E_1\\ F^1{}_1 &= 0\\ F^1{}_2 &= B_3\\ F^1{}_3 &= -B_2\end{aligned} \tag{15}$$

The other components can be found similarly, and we get MTW's eqn 3.5:

$$\left\|F^\alpha{}_\beta\right\| = \begin{bmatrix} 0 & E_1 & E_2 & E_3\\ E_1 & 0 & B_3 & -B_2\\ E_2 & -B_3 & 0 & B_1\\ E_3 & B_2 & -B_1 & 0\end{bmatrix} \tag{16}$$

We can lower the first index using the metric

$$\eta = \begin{bmatrix} -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \tag{17}$$

We get

$$F_{\alpha\beta} = \eta_{\alpha\gamma}F^\gamma{}_\beta \tag{18}$$

Multiplying the matrix 17 into 16 just flips the signs of the first row, and we get

$$\left\|F_{\alpha\beta}\right\| = \begin{bmatrix} 0 & -E_1 & -E_2 & -E_3\\ E_1 & 0 & B_3 & -B_2\\ E_2 & -B_3 & 0 & B_1\\ E_3 & B_2 & -B_1 & 0\end{bmatrix} \tag{19}$$

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