TRANSFORMATION LAW FOR TENSOR COMPONENTS

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References: Charles W. Misner, Kip S. Thorne & John Archibald Wheeler, *Gravitation*, W.H. Freeman (1973). Sections 2.9 & 3.2; Exercise 3.2.

Post date: 19 Jul 2020.

In section 2.9, MTW derive the transformation laws for one-forms and vectors. A 4-vector v can be written in terms of basis vectors $\boldsymbol{e}_\alpha$ as

$$v = v^\alpha \boldsymbol{e}_\alpha \tag{1}$$

Under a Lorentz transformation specified by the matrix $\Lambda^{\alpha'}{}_{\beta}$, the coordinates x^β of an unprimed system transform to coordinates $x^{\alpha'}$ of a primed system:

$$x^{\alpha'} = \Lambda^{\alpha'}{}_{\beta} x^\beta \tag{2}$$

with the inverse transformation from primed to unprimed

$$x^{\alpha} = \Lambda^{\alpha}{}_{\beta'} x^{\beta'} \tag{3}$$

If we insert 2 into 3 we get

$$x^{\alpha} = \Lambda^{\alpha}{}_{\beta'} \Lambda^{\beta'}{}_{\gamma} x^{\gamma} \tag{4}$$

Since the RHS must equal x^α, we find that

$$\Lambda^{\alpha}{}_{\beta'} \Lambda^{\beta'}{}_{\gamma} = \delta^{\alpha}{}_{\gamma} \tag{5}$$

In other words, the transformation from primed to unprimed must be the inverse of the transformation from unprimed to primed (which I guess is fairly obvious).

The 4-velocity u is given by

$$u \equiv \frac{dx^\alpha}{d\tau} \boldsymbol{e}_\alpha \tag{6}$$

This must be independent of the coordinate system, so we must also have

$$u = \frac{dx^{\alpha'}}{d\tau}\boldsymbol{e}_{\alpha'} \tag{7}$$

$$= \Lambda^{\alpha'}{}_{\beta}\frac{dx^{\beta}}{d\tau}\boldsymbol{e}_{\alpha'} \tag{8}$$

$$= \frac{dx^{\beta}}{d\tau}\boldsymbol{e}_{\beta} \tag{9}$$

From this, we get the transformation law for the basis vectors:

$$\boldsymbol{e}_{\beta} = \Lambda^{\alpha'}{}_{\beta}\boldsymbol{e}_{\alpha'} \tag{10}$$

and its inverse

$$\boldsymbol{e}_{\beta'} = \Lambda^{\alpha}{}_{\beta'}\boldsymbol{e}_{\alpha} \tag{11}$$

We can use similar derivations to get the transformation laws for general vector and one-form components, given as MTW's eqns 2.41 - 2.43 which we summarize here.

$$\begin{aligned}
\boldsymbol{e}_{\alpha'} &= \Lambda^{\beta}{}_{\alpha'}\boldsymbol{e}_{\beta} \\
\boldsymbol{e}_{\alpha} &= \Lambda^{\beta'}{}_{\alpha}\boldsymbol{e}_{\beta'} \\
\boldsymbol{\omega}^{\alpha'} &= \Lambda^{\alpha'}{}_{\beta}\boldsymbol{\omega}^{\beta} \\
\boldsymbol{\omega}^{\alpha} &= \Lambda^{\alpha}{}_{\beta'}\boldsymbol{\omega}^{\beta'} \\
v^{\alpha'} &= \Lambda^{\alpha'}{}_{\beta}v^{\beta} \\
v^{\alpha} &= \Lambda^{\alpha}{}_{\beta'}v^{\beta'} \\
\sigma_{\alpha'} &= \Lambda^{\beta}{}_{\alpha'}\sigma_{\beta} \\
\sigma_{\alpha} &= \Lambda^{\beta'}{}_{\alpha}\sigma_{\beta'}
\end{aligned} \tag{12}$$

where $\boldsymbol{\omega}^{\alpha}$ is a basis one-form in the unprimed system, v^{α} is a component of the vector v in the unprimed system, and σ_{α} is a component of the one-form σ in the unprimed system; analogous symbols are used for the primed system.

Ex. 3.2. In section 3.2, MTW define a general tensor as an object which takes one or more vectors and/or one-forms and whose output is either a scalar or a vector. They define a general tensor H to have rank $\binom{n}{m}$ if it takes n one-forms and m vectors as its arguments. In a given Lorentz frame, we can find the basis components of a general tensor S by inserting the basis one-forms and vectors as its arguments. Thus for a $\binom{2}{1}$ tensor that takes 2 one-forms and 1 vector as arguments, its basis components are given by

$$S^{\alpha\beta}{}_{\gamma} \equiv S\left(\boldsymbol{\omega}^{\alpha},\boldsymbol{\omega}^{\beta},\boldsymbol{e}_{\gamma}\right) \tag{13}$$

Because tensors are linear in their arguments, we can use this to build up the components of any tensor in a given Lorentz frame. This is shown in MTW's eqn 3.13:

$$S(\boldsymbol{\sigma},\boldsymbol{\rho},\boldsymbol{v}) = S^{\alpha\beta}{}_{\gamma}\sigma_{\alpha}\rho_{\beta}v^{\gamma} \tag{14}$$

We can now derive the transformation law for a general tensor. The tensor object $S(\boldsymbol{\sigma},\boldsymbol{\rho},\boldsymbol{v})$ is coordinate independent, so we must have

$$S^{\delta'\epsilon'}{}_{\zeta'}\sigma_{\epsilon'}\rho_{\beta'}v^{\zeta'} = S^{\alpha\beta}{}_{\gamma}\sigma_{\alpha}\rho_{\beta}v^{\gamma} \tag{15}$$

Insert the appropriate transformations from 12 into 14 and we have

$$S^{\alpha\beta}{}_{\gamma}\sigma_{\alpha}\rho_{\beta}v^{\gamma} = S^{\alpha\beta}{}_{\gamma}\Lambda^{\delta'}{}_{\alpha}\sigma_{\delta'}\Lambda^{\epsilon'}{}_{\beta}\rho_{\epsilon'}\Lambda^{\gamma}{}_{\zeta'}v^{\zeta'} \tag{16}$$

$$= S^{\alpha\beta}{}_{\gamma}\Lambda^{\delta'}{}_{\alpha}\Lambda^{\epsilon'}{}_{\beta}\Lambda^{\gamma}{}_{\zeta'}\sigma_{\delta'}\rho_{\epsilon'}v^{\zeta'} \tag{17}$$

Comparing with 15 we have

$$S^{\delta'\epsilon'}{}_{\zeta'} = S^{\alpha\beta}{}_{\gamma}\Lambda^{\delta'}{}_{\alpha}\Lambda^{\epsilon'}{}_{\beta}\Lambda^{\gamma}{}_{\zeta'} \tag{18}$$

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