RAISING AND LOWERING TENSOR INDICES

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References: Charles W. Misner, Kip S. Thorne & John Archibald Wheeler, *Gravitation*, W.H. Freeman (1973). Sections 2.9 & 3.2; Exercise 3.3.

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A general tensor S is a function of a number of one-forms and/or vectors which outputs either a scalar or vector. In a given Lorentz frame, the components of a tensor are found by the action of the tensor on the basis one-forms and vectors.

$$S(\boldsymbol{\sigma},\boldsymbol{\rho},\boldsymbol{v}) = S^{\alpha\beta}{}_{\gamma}\sigma_\alpha\rho_\beta v^\gamma \tag{1}$$

where $\boldsymbol{\sigma}$ and $\boldsymbol{\rho}$ are one-forms and $\boldsymbol{v}$ is a vector.

The components of a vector can be converted to the components of the corresponding one-form, and vice versa, using the metric tensor η in an operation known as raising or lowering the indices. The components of a one-form have lower indices which are raised by the operation

$$\sigma^\alpha = \eta^{\alpha\beta}\sigma_\beta \tag{2}$$

so that σ^α are the components of the vector $\boldsymbol{\sigma}$ corresponding to the one-form $\boldsymbol{\sigma}$. MTW use the same symbol for both the one-form and the vector as they are equivalent in the sense that they contain the same information.

The components of a vector have upper indices which can be lowered by

$$v_\alpha = \eta_{\alpha\beta}v^\alpha \tag{3}$$

to convert a vector into its corresponding one-form.

The tensor S itself, as a function of its one-form and vector arguments, is independent of whether its arguments are expressed as one-forms or vectors. That is, we require

$$S^{\alpha\beta}{}_{\gamma}\sigma_\alpha\rho_\beta v^\gamma = S^{\alpha}{}_{\beta\gamma}\sigma_\alpha\rho^\beta v^\gamma \tag{4}$$

$$= S^{\alpha\beta\gamma}\sigma_\alpha\rho_\beta v_\gamma \tag{5}$$

$$= S^{\alpha}{}_{\beta}{}^{\gamma}\sigma_\alpha\rho^\beta v_\gamma \tag{6}$$

and so on for all other possible positions of the indices α, β and γ.

Ex. 3.3. Using 2 and 3 we can relate the various forms of tensor components to each other. For example, we have

$$S^{\alpha\beta}{}_{\gamma}\sigma_{\alpha}\rho_{\beta}v^{\gamma} = S^{\alpha}{}_{\beta\gamma}\sigma_{\alpha}\rho^{\beta}v^{\gamma} \tag{7}$$
$$= S^{\alpha}{}_{\beta\gamma}\sigma_{\alpha}\eta^{\beta\delta}\rho_{\delta}v^{\gamma} \tag{8}$$
$$= \eta^{\beta\delta}S^{\alpha}{}_{\beta\gamma}\sigma_{\alpha}\rho_{\delta}v^{\gamma} \tag{9}$$
$$= \eta^{\delta\beta}S^{\alpha}{}_{\delta\gamma}\sigma_{\alpha}\rho_{\beta}v^{\gamma} \tag{10}$$

where in the last line we swapped the dummy indices $\beta \leftrightarrow \delta$, which we can do because they are summed. Comparing the LHS and RHS, we see that, since this relation must be true for all vectors and one-forms, we must have

$$S^{\alpha\beta}{}_{\gamma} = \eta^{\delta\beta}S^{\alpha}{}_{\delta\gamma} \tag{11}$$

That is, we can raise an index of a tensor components by using the metric η. A similar argument shows that we can lower an index in the same way, or we can use the fact that the inverse of the metric $\eta^{\delta\beta}$ is the same metric with lower indices:

$$\left\|\eta_{\alpha\beta}\right\| = \left\|\eta^{\alpha\beta}\right\|^{-1} \tag{12}$$

Multiplying 11 both sides by $\eta_{\beta\epsilon}$ and summing, we get

$$\eta_{\beta\epsilon}S^{\alpha\beta}{}_{\gamma} = \eta_{\beta\epsilon}\eta^{\delta\beta}S^{\alpha}{}_{\delta\gamma} \tag{13}$$
$$= \delta^{\delta}{}_{\epsilon}S^{\alpha}{}_{\delta\gamma} \tag{14}$$
$$= S^{\alpha}{}_{\epsilon\gamma} \tag{15}$$

Thus we've lowered the β index using the lower form of the metric $\eta_{\beta\epsilon}$.

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