# TENSOR PRODUCT OF VECTORS AND ONE-FORMS

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References: Charles W. Misner, Kip S. Thorne & John Archibald Wheeler, *Gravitation*, W.H. Freeman (1973). Exercises 3.4-3.5.

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The tensor product of two vectors $u$ and $v$ is defined by MTW as

$$(u \otimes v)(\sigma,\lambda) \equiv \langle\sigma,u\rangle\langle\lambda,v\rangle \tag{1}$$

Notice that the tensor product is defined as a function of two one-forms $\sigma$ and $\lambda$, and the result is the product of the inner products of the one-forms with the vectors. In components, we have

$$(u \otimes v)(\sigma,\lambda) = \sigma_\mu u^\mu \lambda_\nu v^\nu \tag{2}$$

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Ex 3.4. If we define

$$T \equiv u \otimes v \tag{3}$$

we can work out the components of $T$. The components of a tensor are found by calculating its values when it operates on basis vectors or one-forms. Therefore, we have

$$T^{\alpha\beta} = (u \otimes v)\left(\boldsymbol{\omega}^\alpha,\boldsymbol{\omega}^\beta\right) \tag{4}$$

where $\boldsymbol{\omega}^\alpha$ represents the basis one-forms. We have

$$T^{\alpha\beta} = \langle\boldsymbol{\omega}^\alpha,u\rangle\left\langle\boldsymbol{\omega}^\beta,v\right\rangle \tag{5}$$

$$= u^\alpha v^\beta \tag{6}$$

because

$$\langle\boldsymbol{\omega}^\alpha,u\rangle = u^\alpha \tag{7}$$

From here, we can use the formulas for raising and lowering indices to get

$$T_{\alpha}{}^{\beta} = \eta_{\alpha\gamma}T^{\gamma\beta} \tag{8}$$
$$= \eta_{\alpha\gamma}u^{\gamma}v^{\beta} \tag{9}$$
$$= u_{\alpha}v^{\beta} \tag{10}$$

where $u_{\alpha}$ is a component of the one-form corresponding to the original vector $u$. Likewise, we have

$$T_{\alpha\beta} = \eta_{\beta\gamma}T_{\alpha}{}^{\gamma} \tag{11}$$
$$= \eta_{\beta\gamma}u_{\alpha}v^{\gamma} \tag{12}$$
$$= u_{\alpha}v_{\gamma} \tag{13}$$

The definition 1 can be extended to include products of vectors with one-forms. For example, if $u$ is a vector and $\lambda$ is a one-form, we have

$$(u\otimes\lambda)(\sigma,v) \equiv \langle\sigma,u\rangle\langle\lambda,v\rangle \tag{14}$$

Note that the RHSs of 1 and 14 are the same; we've merely interchanged the locations of $\lambda$ and $v$ on the LHS. The important point is that if one of the objects in the tensor product is a vector, then it must be contracted with a one-form, and vice versa.

Using basis one-forms and vectors, we can find the components of $R = u\otimes\lambda$:

$$R^{\alpha}{}_{\beta} = (u\otimes\lambda)\left(\boldsymbol{\omega}^{\alpha},\boldsymbol{e}_{\beta}\right) \tag{15}$$
$$= u^{\alpha}\lambda_{\beta} \tag{16}$$

with other components with raised or lowered indices found in the usual way by using the metric $\eta$.

We can extend the tensor product to contain more than two objects, as in

$$S = u\otimes v\otimes\beta\otimes w(\sigma,\lambda,n,\zeta) = \langle\sigma,u\rangle\langle\lambda,v\rangle\langle\beta,n\rangle\langle\zeta,w\rangle \tag{17}$$

Again, note that a one-form is always paired with a vector.

We can again work out the components by operating on unit one-forms and vectors. Thus

$$S^{\mu\nu}{}_{\lambda}{}^{\zeta} = u\otimes v\otimes\beta\otimes w\left(\boldsymbol{\omega}^{\mu},\boldsymbol{\omega}^{\nu},\boldsymbol{e}_{\lambda},\boldsymbol{\omega}^{\zeta}\right) \tag{18}$$
$$= \langle\boldsymbol{\omega}^{\mu},u\rangle\langle\boldsymbol{\omega}^{\nu},v\rangle\langle\beta,\boldsymbol{e}_{\lambda}\rangle\left\langle\boldsymbol{\omega}^{\zeta},w\right\rangle \tag{19}$$
$$= u^{\mu}v^{\nu}\beta_{\lambda}w^{\zeta} \tag{20}$$

Note that all these manipulations with components apply to a particular Lorentz frame, since the basis vectors and one-forms apply in only one particular Lorentz frame.

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Ex 3.5. Now suppose we know the components $M^{\alpha\beta}{}_{\gamma}{}^{\delta}$ of a tensor in a particular Lorentz frame. We can multiply these components by the tensor product of the basis vectors and one-forms in that frame, as follows.

$$M = M^{\alpha\beta}{}_{\gamma}{}^{\delta}\boldsymbol{e}_{\alpha}\otimes\boldsymbol{e}_{\beta}\otimes\boldsymbol{\omega}^{\gamma}\otimes\boldsymbol{e}_{\delta} \tag{21}$$

with sums implied over all 4 pairs of repeated indices. Suppose this tensor $M$ is composed of the tensor product of 3 vectors $u$, $v$ and $w$, and a one-form $\sigma$. Then its components can be written as

$$M^{\alpha\beta}{}_{\gamma}{}^{\delta} = \langle\boldsymbol{\omega}^{\alpha},u\rangle\left\langle\boldsymbol{\omega}^{\beta},v\right\rangle\langle\sigma,\boldsymbol{e}_{\gamma}\rangle\left\langle\boldsymbol{\omega}^{\delta},w\right\rangle \tag{22}$$

$$= u^{\alpha}v^{\beta}\sigma_{\gamma}w^{\delta} \tag{23}$$

Substituting this into 21 we have

$$M = u^{\alpha}v^{\beta}\sigma_{\gamma}w^{\delta}\left(\boldsymbol{e}_{\alpha}\otimes\boldsymbol{e}_{\beta}\otimes\boldsymbol{\omega}^{\gamma}\otimes\boldsymbol{e}_{\delta}\right) \tag{24}$$

$$= (u^{\alpha}\boldsymbol{e}_{\alpha})\otimes\left(v^{\beta}\boldsymbol{e}_{\beta}\right)\otimes(\sigma_{\gamma}\boldsymbol{\omega}^{\gamma})\otimes\left(w^{\delta}\boldsymbol{e}_{\delta}\right) \tag{25}$$

$$= u\otimes v\otimes\sigma\otimes w \tag{26}$$

where the last line follows because a vector such as $u$ can be written in terms of the basis vectors as

$$u = u^{\alpha}\boldsymbol{e}_{\alpha} \tag{27}$$

and similarly for one-forms. Thus, given 21, we can reconstruct the original tensor product. Note that, although the components $M^{\alpha\beta}{}_{\gamma}{}^{\delta}$ and basis vectors and basis one-forms are given in a particular Lorentz frame, the tensor itself 26 makes no reference to components or basis vectors and is thus a frame-independent representation of the tensor product.

## PINGBACKS

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