ELECTROMAGNETIC FIELD (FARADAY) TENSOR AT WORK

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References: Charles W. Misner, Kip S. Thorne & John Archibald Wheeler, *Gravitation*, W.H. Freeman (1973). Exercise 3.6.

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The electromagnetic field tensor, or Faraday tensor as MTW call it, is

$$\left\|F^{\alpha}{}_{\beta}\right\| = \begin{bmatrix} 0 & E_1 & E_2 & E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{bmatrix} \tag{1}$$

We can define the tensor in a coordinate-free environment so that it is a function of a single vector u and returns a vector $F(u)$, or we can introduce a one-form σ and define F so that it is a function of both σ and u as in

$$F(\sigma,u) = \langle\sigma, F(u)\rangle \tag{2}$$

In components, these two interpretations read

$$F(u) = F^{\alpha}{}_{\beta}u^{\beta} \tag{3}$$

$$F(\sigma,u) = \langle\sigma, F(u)\rangle \tag{4}$$

$$= \sigma_{\alpha}F^{\alpha}{}_{\beta}u^{\beta} \tag{5}$$

As always, the components are specific to a Lorentz frame.

Ex 3.6. We are given an observer with 4-velocity u who defines three spatial unit vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ and $\boldsymbol{e}_3$ that are mutually orthonormal and define a right-handed coordinate system. In the observer's frame, we have

$$\begin{aligned} u &= (1,0,0,0) \\ \boldsymbol{e}_1 &= (0,1,0,0) \\ \boldsymbol{e}_2 &= (0,0,1,0) \\ \boldsymbol{e}_3 &= (0,0,0,1) \end{aligned} \tag{6}$$

Thus in this frame

$$u\cdot\boldsymbol{e}_i = 0 \tag{7}$$

$$\boldsymbol{e}_i\cdot\boldsymbol{e}_j = \delta_{ij} \tag{8}$$

Since these are scalar products, they are valid in any Lorentz frame.

We now want to find what one-form and what vector we need to insert into 2 to obtain the electric and magnetic fields as seen by the observer in his frame. Since we're in the observer's frame, we can use the vectors as in 6. For the electric field we have

$$\begin{aligned}\langle u, F(\boldsymbol{e}_i)\rangle &= u_\mu F^\mu{}_\nu \boldsymbol{e}_i^\nu &(9)\\ &= -F^0{}_i &(10)\\ &= -E_i &(11)\end{aligned}$$

The minus sign appears because MTW's metric is

$$\eta = \begin{bmatrix} -1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \tag{12}$$

so when we lower the $u^0 = 1$ component of u, we get $u_0 = -1$.

For the magnetic field, let us choose $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ to insert into 2. We get

$$\begin{aligned}\langle \boldsymbol{e}_1, F(\boldsymbol{e}_2)\rangle &= (\boldsymbol{e}_1)_\mu F^\mu{}_\nu \boldsymbol{e}_2^\nu &(13)\\ &= F^1{}_2 &(14)\\ &= B_3 &(15)\end{aligned}$$

In this case, there is no sign change when we lower the index from $\boldsymbol{e}_1^\mu$ to $(\boldsymbol{e}_1)_\mu$ since the only non-zero component is a spatial component.

We can generalize this result to see that

$$\langle \boldsymbol{e}_i, F(\boldsymbol{e}_j)\rangle = \varepsilon_{ijk} B_k \tag{16}$$

where ε_{ijk} is the usual anti-symmetric symbol that satisfies

$$\varepsilon_{ijk} = \begin{cases} 1 & \text{if } ijk \text{ is an even permutation of 123}\\ -1 & \text{if } ijk \text{ is an odd permutation of 123}\\ 0 & \text{if any 2 of } ijk \text{ are equal}\end{cases} \tag{17}$$

For example,

$$\varepsilon_{123} = \varepsilon_{312} = \varepsilon_{231} = +1 \tag{18}$$
$$\varepsilon_{213} = \varepsilon_{321} = \varepsilon_{132} = -1 \tag{19}$$
$$\varepsilon_{112} = \ldots = 0 \tag{20}$$

PINGBACKS

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