CONTRACTION OF A TENSOR AND INDEPENDENCE OF LORENTZ FRAME

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References: Charles W. Misner, Kip S. Thorne & John Archibald Wheeler, *Gravitation*, W.H. Freeman (1973). Exercise 3.8.

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MTW define the contraction of a tensor in a coordinate-free form as

$$M(u,v) \equiv \sum_{\alpha=0}^{3} R(\boldsymbol{e}_\alpha, u, \boldsymbol{\omega}^\alpha, v) \tag{1}$$

where $\boldsymbol{e}_\alpha$ are the basis vectors and $\boldsymbol{\omega}^\alpha$ are the basis one-forms. It might appear that this definition does depend on the reference frame, since the basis vectors and one-forms are defined for a specific frame. However, we can see that this definition is in fact independent of the Lorentz frame. We can see this by using the transformation laws for the basis vectors and one-forms. The relevant laws are

$$\begin{aligned} \boldsymbol{e}_{\alpha'} &= \Lambda^\beta{}_{\alpha'} \boldsymbol{e}_\beta \\ \boldsymbol{\omega}^{\alpha'} &= \Lambda^{\alpha'}{}_\beta \boldsymbol{\omega}^\beta \end{aligned} \tag{2}$$

Consider 1 evaluated in the primed frame. We have (I'll use the summation convention where all pairs of lower and upper indices are summed):

$$R\left(\boldsymbol{e}_{\alpha'}, u, \boldsymbol{\omega}^{\alpha'}, v\right) = R\left(\Lambda^\beta{}_{\alpha'} \boldsymbol{e}_\beta, u, \Lambda^{\alpha'}{}_\gamma \boldsymbol{\omega}^\gamma, v\right) \tag{3}$$

$$= \Lambda^\beta{}_{\alpha'} \Lambda^{\alpha'}{}_\gamma R\left(\boldsymbol{e}_\beta, u, \boldsymbol{\omega}^\gamma, v\right) \tag{4}$$

We now use the property of the Lorentz transformation matrix (MTW eqn. 2.39):

$$\Lambda^\beta{}_{\alpha'} \Lambda^{\alpha'}{}_\gamma = \delta^\beta{}_\gamma \tag{5}$$

so we have

$$R\left(\boldsymbol{e}_{\alpha'}, u, \boldsymbol{\omega}^{\alpha'}, v\right) = \delta^\beta{}_\gamma R\left(\boldsymbol{e}_\beta, u, \boldsymbol{\omega}^\gamma, v\right) \tag{6}$$

$$= R\left(\boldsymbol{e}_\beta, u, \boldsymbol{\omega}^\beta, v\right) \tag{7}$$

which is the same as 1 since we can relabel the dummy β index to α.

We can also find a component representation of $M(u,v)$ in 1. We have

$$R(\boldsymbol{e}_\alpha, u, \boldsymbol{\omega}^\alpha, v) = R\left(\boldsymbol{e}_\alpha, u^\beta \boldsymbol{e}_\beta, \boldsymbol{\omega}^\alpha, v^\gamma \boldsymbol{e}_\gamma\right) \tag{8}$$

$$= u^\beta v^\gamma R\left(\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta, \boldsymbol{\omega}^\alpha, \boldsymbol{e}_\gamma\right) \tag{9}$$

$$= u^\beta v^\gamma R_{\alpha\beta}{}^{\alpha}{}_{\gamma} \tag{10}$$

where to get the last line, we've used the definition of the components of a tensor in a specific Lorentz frame:

$$R_{\alpha\beta}{}^{\alpha}{}_{\gamma} = R\left(\boldsymbol{e}_\alpha, \boldsymbol{e}_\beta, \boldsymbol{\omega}^\alpha, \boldsymbol{e}_\gamma\right) \tag{11}$$

Therefore

$$M(u,v) = R_{\alpha\beta}{}^{\alpha}{}_{\gamma} u^\beta v^\gamma \tag{12}$$