TENSOR DIFFERENTIATION

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References: Charles W. Misner, Kip S. Thorne & John Archibald Wheeler, *Gravitation*, W.H. Freeman (1973). Exercises 3.9 - 3.10.

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Ex 3.9. (a) We're asked to justify the formula

$$\frac{d}{d\tau}\left(u_{\mu}v^{\nu}\right)=\frac{du_{\mu}}{d\tau}v^{\nu}+u_{\mu}\frac{dv^{\nu}}{d\tau} \tag{1}$$

Unless I'm missing something, this would appear to be a straightforward application of the product rule for derivatives. Since μ and ν are different indices, no summation is implied.

(b) Now for a formula with an implied sum. MTW indicate a derivative with respect to a spacetime coordinate x^{μ} using the notation

$$\frac{\partial T^{\alpha\beta}}{\partial x^{\mu}}\equiv T^{\alpha\beta}{}_{,\mu} \tag{2}$$

We have

$$\left(T^{\alpha\beta}v_{\beta}\right)_{,\mu}=\frac{\partial}{\partial x^{\mu}}\left(T^{\alpha0}v_{0}+T^{\alpha1}v_{1}+T^{\alpha2}v_{2}+T^{\alpha3}v_{3}\right) \tag{3}$$

$$=v_{0}\frac{\partial}{\partial x^{\mu}}T^{\alpha0}+T^{\alpha0}\frac{\partial v_{0}}{\partial x^{\mu}}+v_{1}\frac{\partial}{\partial x^{\mu}}T^{\alpha1}+T^{\alpha1}\frac{\partial v_{1}}{\partial x^{\mu}}+ v_{2}\frac{\partial}{\partial x^{\mu}}T^{\alpha2}+T^{\alpha2}\frac{\partial v_{2}}{\partial x^{\mu}}+v_{3}\frac{\partial}{\partial x^{\mu}}T^{\alpha3}+T^{\alpha3}\frac{\partial v_{3}}{\partial x^{\mu}} \tag{4}$$

$$=T^{\alpha\beta}{}_{,\mu}v_{\beta}+T^{\alpha\beta}v_{\beta,\mu} \tag{5}$$

Ex 3.10. (a) The derivative of the inner product of a vector with its corresponding one-form is (implied sums)

$$\frac{d}{d\tau}\left(u^{\mu}u_{\mu}\right)=\frac{d}{d\tau}\left(\eta_{\mu\nu}u^{\mu}u^{\nu}\right) \tag{6}$$
$$=\eta_{\mu\nu}\frac{d}{d\tau}\left(u^{\mu}u^{\nu}\right) \tag{7}$$
$$=\eta_{\mu\nu}u^{\nu}\frac{du^{\mu}}{d\tau}+\eta_{\mu\nu}u^{\mu}\frac{du^{\nu}}{d\tau} \tag{8}$$
$$=2\eta_{\mu\nu}u^{\mu}\frac{du^{\nu}}{d\tau} \tag{9}$$
$$=2u_{\mu}\frac{du^{\nu}}{d\tau} \tag{10}$$

To get from the third to the fourth line, we can swap $\nu \leftrightarrow \mu$ in the first term, since μ and ν are both dummy (summed indices). The elements of the metric $\eta_{\mu\nu}$ are constants, so they are unaffected by the derivative.

(b) For the tensor $F_{\alpha\beta}$, we can consider a small change using the symbol δ and apply the product rule:

$$\delta\left(F_{\alpha\beta}F^{\alpha\beta}\right)=F^{\alpha\beta}\delta F_{\alpha\beta}+F_{\alpha\beta}\delta F^{\alpha\beta} \tag{11}$$
$$=F^{\alpha\beta}\eta_{\alpha\gamma}\eta_{\beta\epsilon}\delta F^{\gamma\epsilon}+F_{\alpha\beta}\delta F^{\alpha\beta} \tag{12}$$
$$=F_{\gamma\epsilon}\delta F^{\gamma\epsilon}+F_{\alpha\beta}\delta F^{\alpha\beta} \tag{13}$$
$$=2F_{\alpha\beta}\delta F^{\alpha\beta} \tag{14}$$

To get the second line, we used

$$\delta F_{\alpha\beta}=\eta_{\alpha\gamma}\eta_{\beta\epsilon}\delta F^{\gamma\epsilon} \tag{15}$$

To get the third line, we used

$$F_{\gamma\epsilon}=F^{\alpha\beta}\eta_{\alpha\gamma}\eta_{\beta\epsilon} \tag{16}$$

and to get the last line, we relabelled the indices $\gamma\to\alpha$ and $\epsilon\to\beta$ in the first term, again because both these indices are summed.

(c) The derivative is calculated the same way, by using the product rule, and we get

$$\left(F_{\alpha\beta}F^{\alpha\beta}\right)_{,\mu}=2F_{\alpha\beta}F^{\alpha\beta}{}_{,\mu} \tag{17}$$