SYMMETRIC AND ANTISYMMETRIC TENSORS

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References: Charles W. Misner, Kip S. Thorne & John Archibald Wheeler, *Gravitation*, W.H. Freeman (1973). Exercises 3.11.-3.12
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A rank-2 tensor is symmetric if

$$S^{\mu\nu}=S^{\nu\mu} \tag{1}$$

and antisymmetric if

$$A^{\mu\nu}=-A^{\nu\mu} \tag{2}$$

Ex 3.11 (a) Taking the product of a symmetric and antisymmetric tensor and summing over all indices gives zero. MTW ask us to show this by writing out all 16 components in the sum. Probably not really needed but for the pendantic among the audience, here goes. First, we note that for an antisymmetric tensor, all diagonal elements must be 0 since $A_{\mu\mu}=-A_{\mu\mu}$ (no sum), so we can omit those from the sum. We get

$$\begin{aligned} A_{\mu\nu}S^{\mu\nu} = {} & A_{01}S^{01}+A_{02}S^{02}+A_{03}S^{03}+ \\ & A_{10}S^{10}+A_{12}S^{12}+A_{13}S^{13}+ \\ & A_{20}S^{20}+A_{21}S^{21}+A_{23}S^{23}+ \\ & A_{30}S^{30}+A_{31}S^{31}+A_{32}S^{32} \end{aligned} \tag{3}$$

Using 1 and 2 we can swap indices in the above to get

$$\begin{aligned} A_{\mu\nu}S^{\mu\nu} = {} & A_{01}S^{01}+A_{02}S^{02}+A_{03}S^{03}- \\ & A_{01}S^{01}+A_{12}S^{12}+A_{13}S^{13}- \\ & A_{02}S^{02}-A_{12}S^{12}+A_{23}S^{23}- \\ & A_{03}S^{03}-A_{13}S^{13}-A_{23}S^{23} \end{aligned} \tag{4}$$

By inspection, the terms cancel in pairs, so we have

$$A_{\mu\nu}S^{\mu\nu}=0 \tag{5}$$

The alternative method given in the question is

$$A_{\mu\nu}S^{\mu\nu} = -A_{\nu\mu}S^{\mu\nu} \tag{6}$$
$$= -A_{\nu\mu}S^{\nu\mu} \tag{7}$$
$$= -A_{\alpha\beta}S^{\alpha\beta} \tag{8}$$
$$= -A_{\mu\nu}S^{\mu\nu} \tag{9}$$

The first line uses 2, the second line uses 1, the third line renames the dummy indices $\nu \to \alpha$ and $\mu \to \beta$, and the fourth line again renames the indices $\alpha \to \mu$ and $\beta \to \nu$. Thus $A_{\mu\nu}S^{\mu\nu} = -A_{\mu\nu}S^{\mu\nu}$ and any quantity equal to its negative must be zero.

(b) We now consider the product of a symmetric or antisymmetric tensor with an arbitrary rank-2 tensor $V_{\mu\nu}$. We have

$$V^{\mu\nu}A_{\mu\nu} = \frac{1}{2}\left(V^{\mu\nu} - V^{\nu\mu}\right)A_{\mu\nu} \tag{10}$$

We can verify this using 2. We have

$$\frac{1}{2}\left(V^{\mu\nu} - V^{\nu\mu}\right)A_{\mu\nu} = \frac{1}{2}\left(V^{\mu\nu}A_{\mu\nu} + V^{\nu\mu}A_{\nu\mu}\right) \tag{11}$$
$$= \frac{1}{2}\left(V^{\mu\nu}A_{\mu\nu} + V^{\mu\nu}A_{\mu\nu}\right) \tag{12}$$
$$= V^{\mu\nu}A_{\mu\nu} \tag{13}$$

In the second line, we swapped $\mu \leftrightarrow \nu$ since they are both summed indices.

We also have, using 1 and swapping $\mu \leftrightarrow \nu$:

$$\frac{1}{2}\left(V^{\mu\nu} + V^{\nu\mu}\right)S_{\mu\nu} = \frac{1}{2}\left(V^{\mu\nu}S_{\mu\nu} + V^{\nu\mu}S_{\nu\mu}\right) \tag{14}$$
$$= \frac{1}{2}\left(V^{\mu\nu}S_{\mu\nu} + V^{\mu\nu}S_{\mu\nu}\right) \tag{15}$$
$$= V^{\mu\nu}S_{\mu\nu} \tag{16}$$

Ex 3.12 We can define a symmetrized tensor for ranks 2 and 3 as

$$V_{(\mu\nu)} \equiv \frac{1}{2}\left(V_{\mu\nu} + V_{\nu\mu}\right) \tag{17}$$
$$V_{(\mu\nu\lambda)} \equiv \frac{1}{3!}\left(V_{\mu\nu\lambda} + V_{\nu\lambda\mu} + V_{\lambda\mu\nu} + V_{\nu\mu\lambda} + V_{\mu\lambda\nu} + V_{\lambda\nu\mu}\right) \tag{18}$$

We also define an antisymmetrized tensor by

$$V_{[\mu\nu]} \equiv \frac{1}{2}\left(V_{\mu\nu} - V_{\nu\mu}\right) \tag{19}$$

$$V_{[\mu\nu\lambda]} \equiv \frac{1}{3!}\left(V_{\mu\nu\lambda} + V_{\nu\lambda\mu} + V_{\lambda\mu\nu} - V_{\nu\mu\lambda} - V_{\mu\lambda\nu} - V_{\lambda\nu\mu}\right) \tag{20}$$

For the rank-3 tensors, note that the first 3 terms on the RHS have even permutations of the 3 indices and the last 3 terms have odd permutations. Note that, for the symmetrized tensors, swapping any two indices results in the same tensor back again. For the antisymmetrized tensors, swapping any two indices results in the negative of the original tensor. For the antisymmetrized tensor, doing 2 swaps gives us the original tensor back again, since positive terms are swapped into positive terms, and negative terms into negative terms.

(a) When inserting vectors into the slots of the rank-3 tensors, we get

$$V_{(\alpha\beta\gamma)}u^{\alpha}v^{\beta}w^{\gamma} \tag{21}$$

Using the symmetry, we have

$$V_{(\alpha\beta\gamma)}u^{\alpha}v^{\beta}w^{\gamma} = V_{(\beta\alpha\gamma)}u^{\alpha}v^{\beta}w^{\gamma} \tag{22}$$

$$= V_{(\beta\alpha\gamma)}v^{\beta}u^{\alpha}w^{\gamma} \tag{23}$$

$$= V_{(\alpha\beta\gamma)}v^{\alpha}u^{\beta}w^{\gamma} \tag{24}$$

To get the last line, we swapped the summed indices $\alpha \leftrightarrow \beta$. The same process works for swapping any two vectors.

In the antisymmetric case, we have, using the same techniques

$$V_{[\alpha\beta\gamma]}u^{\alpha}v^{\beta}w^{\gamma} = -V_{[\beta\alpha\gamma]}u^{\alpha}v^{\beta}w^{\gamma} \tag{25}$$

$$= -V_{[\beta\alpha\gamma]}v^{\beta}u^{\alpha}w^{\gamma} \tag{26}$$

$$= -V_{[\alpha\beta\gamma]}v^{\alpha}u^{\beta}w^{\gamma} \tag{27}$$

(b) From 17 and 19 we have

$$V_{(\mu\nu)} + V_{[\mu\nu]} = \frac{1}{2}\left(V_{\mu\nu} + V_{\nu\mu}\right) + \frac{1}{2}\left(V_{\mu\nu} - V_{\nu\mu}\right) \tag{28}$$

$$= V_{\mu\nu} \tag{29}$$

Thus we can reconstruct a rank-2 tensor from its symmetrized and antisymmetrized versions. The same technique doesn't work for a rank-3 tensor. The reason is that, from 18 and 20, we have only 2 equations to solve for 6 unknowns. Adding $V_{(\mu\nu\lambda)}+V_{[\mu\nu\lambda]}$ eliminates 3 components, but all we get is an equation giving the sum of the other three components.

(c) The electromagnetic field tensor is

$$\|F_{\alpha\beta}\|=\begin{bmatrix}0 & -E_1 & -E_2 & -E_3\\ E_1 & 0 & B_3 & -B_2\\ E_2 & -B_3 & 0 & B_1\\ E_3 & B_2 & -B_1 & 0\end{bmatrix} \tag{30}$$

By inspection, we see that it is already antisymmetric, so

$$F_{\alpha\beta}=F_{[\alpha\beta]} \tag{31}$$

Or, explicitly from 19,

$$F_{[\alpha\beta]}=\frac{1}{2}\left(F_{\alpha\beta}-F_{\beta\alpha}\right) \tag{32}$$

$$=\frac{1}{2}\left(F_{\alpha\beta}+F_{\alpha\beta}\right) \tag{33}$$

$$=F_{\alpha\beta} \tag{34}$$

From 17 we have

$$F_{(\alpha\beta)}=\frac{1}{2}\left(F_{\alpha\beta}+F_{\beta\alpha}\right) \tag{35}$$

$$=\frac{1}{2}\left(F_{\alpha\beta}-F_{\alpha\beta}\right) \tag{36}$$

$$=0 \tag{37}$$

(d) Maxwell's equations are written in terms of $F_{\alpha\beta}$ as

$$F_{\alpha\beta,\gamma}+F_{\beta\gamma,\alpha}+F_{\gamma\alpha,\beta}=0 \tag{38}$$

From 20, we have

$$F_{[\alpha\beta,\gamma]}=\frac{1}{3!}\left(F_{\alpha\beta,\gamma}+F_{\beta\gamma,\alpha}+F_{\gamma\alpha,\beta}-F_{\beta\alpha,\gamma}-F_{\alpha\gamma,\beta}-F_{\gamma\beta,\alpha}\right) \tag{39}$$

Using the antisymmetry of $F_{\alpha\beta}$ to swap the first two indices in each of the last 3 terms, this becomes

$$F_{[\alpha\beta,\gamma]}=\frac{1}{3!}\left(F_{\alpha\beta,\gamma}+F_{\beta\gamma,\alpha}+F_{\gamma\alpha,\beta}+F_{\alpha\beta,\gamma}+F_{\gamma\alpha,\beta}+F_{\beta\gamma,\alpha}\right) \tag{40}$$

$$=\frac{1}{3}\left(F_{\alpha\beta,\gamma}+F_{\beta\gamma,\alpha}+F_{\gamma\alpha,\beta}\right) \tag{41}$$

Note that we can't swap the third index with either of the other two, since it indicates a derivative and not a component of F.

Therefore, 38 is equivalent to writing

$$F_{[\alpha\beta,\gamma]} = 0 \tag{42}$$

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