LEVI-CIVITA TENSOR

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References: Charles W. Misner, Kip S. Thorne & John Archibald Wheeler, *Gravitation*, W.H. Freeman (1973). Exercise 3.13

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MTW define the Levi-Civita tensor of rank 4 in terms of the four unit vectors of a proper Lorentz frame as

$$\varepsilon\left(\boldsymbol{e}_0,\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3\right)=+1 \tag{1}$$

A proper Lorentz transformation matrix is one with determinant +1, has $\boldsymbol{e}_0$ pointing towards the future and with the 3 spatial unit vectors forming a right-handed coordinate system. The tensor's components are then defined by requiring that a component changes sign whenever two of the vectors are swapped.

Ex. 3.13 (a) The components of a tensor in a particular Lorentz frame are found from its action on the unit vectors in that frame. Therefore, from 1, we have

$$\varepsilon_{0123}=+1 \tag{2}$$

The requirement of antisymmetry means that if we swap any two indices, we reverse the sign. If two indices are the same, then swapping them gives the same component back again, but also reverses its sign. That is

$$\varepsilon_{\alpha\alpha\beta\gamma}=-\varepsilon_{\alpha\alpha\beta\gamma} \tag{3}$$

which implies that, if any two indices are equal

$$\varepsilon_{\alpha\beta\gamma\delta}=0 \tag{4}$$

Also, since swapping any two indices reverses the sign, then an even number of swaps (which is an even permutation) will always return the component back to its original sign and an odd number of swaps (an odd permutation) will always reverse the sign. That is

$$\varepsilon_{\pi_0\pi_1\pi_2\pi_3}=\begin{cases}+1 & \text{for even permutations of } 0,1,2,3\\ -1 & \text{for odd permutations of } 0,1,2,3\end{cases} \tag{5}$$

(b) We can raise all 4 indices by multiplying 4 times by the metric

$$\eta = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{6}$$

That is,

$$\varepsilon^{\pi_0\pi_1\pi_2\pi_3} = \eta^{\pi_0\alpha}\eta^{\pi_1\beta}\eta^{\pi_2\gamma}\eta^{\pi_3\delta}\varepsilon_{\alpha\beta\gamma\delta} \tag{7}$$

The indices $\alpha\beta\gamma\delta$ must contain one each of 0, 1, 2, 3, so the product on the RHS 7 will contain one -1 and three $+1$ factors, which means that raising all four indices always changes the sign of a component. That is

$$\varepsilon^{\pi_0\pi_1\pi_2\pi_3} = -\varepsilon_{\pi_0\pi_1\pi_2\pi_3} \tag{8}$$

(c) To show that ε is invariant under Lorentz transformations, we first observe that we can write the determinant of any 4×4 matrix as

$$\det M = \varepsilon_{\alpha\beta\gamma\delta}M^{\alpha 0}M^{\beta 1}M^{\gamma 2}M^{\delta 3} \tag{9}$$

This is a theorem from matrix algebra, but we've done a derivation for a 3×3 matrix earlier. We can lower the second index on each matrix component to get

$$\det M = \varepsilon_{\alpha\beta\gamma\delta}M^{\alpha}{}_{0}M^{\beta}{}_{1}M^{\gamma}{}_{2}M^{\delta}{}_{3} \tag{10}$$

We can now use the theorem that swapping any two columns or rows of a matrix changes the sign of its determinant to write

$$\varepsilon_{\rho\sigma\tau\upsilon}\det M = \varepsilon_{\alpha\beta\gamma\delta}M^{\alpha}{}_{\rho}M^{\beta}{}_{\sigma}M^{\gamma}{}_{\tau}M^{\delta}{}_{\upsilon} \tag{11}$$

Note that if $\rho\sigma\tau\upsilon = 0123$,this reduces to 10, and that the $\varepsilon_{\rho\sigma\tau\upsilon}$ factor on the LHS provides the sign switch when two columns of M are swapped.

Since 11 is true for all 4×4 matrices, it is true for the Lorentz transformation matrix Λ. From the condition

$$\Lambda^T\eta\Lambda = \eta \tag{12}$$

we can take the determinant of both sides and use the facts that the determinant of a product is the product of the determinants, and that the determinant of a transpose is equal to the determinant of the original, to see that

$$\det\left(\Lambda^T\eta\Lambda\right) = (\det\Lambda)^2\det\eta = -(\det\Lambda)^2 = \det\eta = -1 \tag{13}$$

Therefore

$$\det\Lambda=\pm1 \tag{14}$$

For the usual proper Lorentz transformation, $\det\Lambda=+1$. We can verify this for the simple case where we have a boost along the x axis, since there we have

$$\Lambda=\begin{bmatrix}\gamma & -\beta\gamma & 0 & 0\\ -\beta\gamma & \gamma & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \tag{15}$$

The determinant is

$$\det\Lambda=\gamma^2\left(1-\beta^2\right) \tag{16}$$

$$=\frac{1-\beta^2}{1-\beta^2} \tag{17}$$

$$=1 \tag{18}$$

I imagine we could do a proof for a general Lorentz transformation, although this would be very messy to do by brute force. I've not been able to find any easier way of doing it, so if you know of one, do leave a comment (see link at the top of the post).

Anyway, returning to 11 with $M=\Lambda$ and $\det\Lambda=1$, we have

$$\varepsilon_{\rho\sigma\tau\upsilon}=\varepsilon_{\alpha\beta\gamma\delta}\Lambda^{\alpha}{}_{\rho}\Lambda^{\beta}{}_{\sigma}\Lambda^{\gamma}{}_{\tau}\Lambda^{\delta}{}_{\upsilon} \tag{19}$$

which shows that ε is invariant under a proper Lorentz transformation.

(d) From the definition 1, we can find the effect of using a Lorentz frame where the time unit vector points towards the past. In this case we replace $\boldsymbol{e}_0$ in 1 by $-\boldsymbol{e}_0$ so we have

$$\varepsilon_{0123}=\varepsilon\left(-\boldsymbol{e}_0,\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3\right) \tag{20}$$

$$=-\varepsilon\left(\boldsymbol{e}_0,\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3\right) \tag{21}$$

$$=-1 \tag{22}$$

Also, if we keep the time unit vector pointing towards the future but reverse one of the spatial unit vectors so that we have a left-handed spatial coordinate system, then we replace one of $\boldsymbol{e}_1,\boldsymbol{e}_2,\boldsymbol{e}_3$ by its negative, so we also change the sign of the components of ε.

(e) We can define a permutation tensor by

$$\delta^{\alpha\beta\gamma}{}_{\mu\nu\lambda}\equiv-\varepsilon^{\alpha\beta\gamma\rho}\varepsilon_{\mu\nu\lambda\rho} \tag{23}$$

From 5, we see that

$$\varepsilon_{\alpha\beta\gamma\rho} = \begin{cases} +\varepsilon_{\mu\nu\lambda\rho} & \text{if } \alpha\beta\gamma \text{ is an even permutation of } \mu\nu\lambda \\ -\varepsilon_{\mu\nu\lambda\rho} & \text{if } \alpha\beta\gamma \text{ is an odd permutation of } \mu\nu\lambda \\ 0 & \text{otherwise} \end{cases} \tag{24}$$

where the '0 otherwise' follows since if $\alpha\beta\gamma$ is neither an even nor an odd permutation of $\mu\nu\lambda$, then $\alpha\beta\gamma\rho$ must contain a repeated index and is therefore 0. Further, from 8,

$$\varepsilon^{\alpha\beta\gamma\rho} = -\varepsilon_{\alpha\beta\gamma\rho} \tag{25}$$

so

$$\varepsilon^{\alpha\beta\gamma\rho} = \begin{cases} -\varepsilon_{\mu\nu\lambda\rho} & \text{if } \alpha\beta\gamma \text{ is an even permutation of } \mu\nu\lambda \\ +\varepsilon_{\mu\nu\lambda\rho} & \text{if } \alpha\beta\gamma \text{ is an odd permutation of } \mu\nu\lambda \\ 0 & \text{otherwise} \end{cases} \tag{26}$$

Finally, from 23

$$\delta^{\alpha\beta\gamma}{}_{\mu\nu\lambda} = \begin{cases} \left(\varepsilon_{\mu\nu\lambda\rho}\right)^2 = +1 & \text{if } \alpha\beta\gamma \text{ is an even permutation of } \mu\nu\lambda \\ -\left(\varepsilon_{\mu\nu\lambda\rho}\right)^2 = -1 & \text{if } \alpha\beta\gamma \text{ is an odd permutation of } \mu\nu\lambda \\ 0 & \text{otherwise} \end{cases} \tag{27}$$

Note that, although there is an implied sum over ρ in the definition 23, there is only one term in this sum that is non-zero, since ρ must be different from all of the other 3 indices. Thus we're justified in writing $\varepsilon^{\alpha\beta\gamma\rho}\varepsilon_{\mu\nu\lambda\rho} = \pm\left(\varepsilon_{\mu\nu\lambda\rho}\right)^2$.

Next, we sum 23 over its third index to get

$$\delta^{\alpha\beta}{}_{\mu\nu} \equiv \frac{1}{2}\delta^{\alpha\beta\gamma}{}_{\mu\nu\lambda} = -\frac{1}{2}\varepsilon^{\alpha\beta\lambda\rho}\varepsilon_{\mu\nu\lambda\rho} \tag{28}$$

Following the same reasoning as above, we have

$$\varepsilon_{\alpha\beta\lambda\rho} = \begin{cases} +\varepsilon_{\mu\nu\lambda\rho} & \text{if } \alpha\beta \text{ is an even permutation of } \mu\nu \\ -\varepsilon_{\mu\nu\lambda\rho} & \text{if } \alpha\beta \text{ is an odd permutation of } \mu\nu \\ 0 & \text{otherwise} \end{cases} \tag{29}$$

so

$$\varepsilon^{\alpha\beta\lambda\rho} = \begin{cases} -\varepsilon_{\mu\nu\lambda\rho} & \text{if } \alpha\beta \text{ is an even permutation of } \mu\nu \\ +\varepsilon_{\mu\nu\lambda\rho} & \text{if } \alpha\beta \text{ is an odd permutation of } \mu\nu \\ 0 & \text{otherwise} \end{cases} \tag{30}$$

In this case, there are 2 non-zero terms in the sum 28. In the first such term, $\lambda\rho$ must be different indices from $\mu\nu$, but we can swap them to get a second term. The swap introduces a sign change, but since we're squaring each term, this sign change cancels out. For $\alpha\beta$ an even permutation of $\mu\nu$ we have (no implied sums):

$$\delta^{\alpha\beta}{}_{\mu\nu} = -\frac{1}{2}\left(-\varepsilon_{\mu\nu\lambda\rho}\varepsilon_{\mu\nu\lambda\rho} - \varepsilon_{\mu\nu\rho\lambda}\varepsilon_{\mu\nu\rho\lambda}\right) \tag{31}$$

$$= -\frac{1}{2}(-1-1) \tag{32}$$

$$= 1 \tag{33}$$

Similarly, for $\alpha\beta$ an odd permutation of $\mu\nu$ we have (no implied sums):

$$\delta^{\alpha\beta}{}_{\mu\nu} = -\frac{1}{2}\left(\varepsilon_{\mu\nu\lambda\rho}\varepsilon_{\mu\nu\lambda\rho} + \varepsilon_{\mu\nu\rho\lambda}\varepsilon_{\mu\nu\rho\lambda}\right) \tag{34}$$

$$= -\frac{1}{2}(1+1) \tag{35}$$

$$= -1 \tag{36}$$

Therefore

$$\delta^{\alpha\beta}{}_{\mu\nu} = \begin{cases} +1 & \text{if } \alpha\beta \text{ is an even permutation of } \mu\nu \\ -1 & \text{if } \alpha\beta \text{ is an odd permutation of } \mu\nu \\ 0 & \text{otherwise} \end{cases} \tag{37}$$

Finally, we have the usual Kronecker delta, in this case defined by

$$\delta^{\alpha}{}_{\mu} \equiv \frac{1}{3}\delta^{\alpha\beta}{}_{\mu\beta} = \frac{1}{6}\delta^{\alpha\beta\lambda}{}_{\mu\beta\lambda} = -\frac{1}{6}\varepsilon^{\alpha\beta\lambda\rho}\varepsilon_{\mu\beta\lambda\rho} \tag{38}$$

If $\alpha = \mu$, there are 3 values of β for which $\delta^{\alpha\beta}{}_{\mu\beta} \neq 0$ and in all 3 cases, $\alpha\beta$ is the same as $\mu\beta$ (and hence is an even permutation), so (no implied sum):

$$\delta^{\alpha}{}_{\alpha} = \frac{1}{3}(1+1+1) = 1 \tag{39}$$

If $\alpha \neq \mu$, then there will be no terms in the sum where $\alpha\beta$ is either an even or an odd permutation of $\mu\beta$, so all terms are zero. For example, if $\alpha = 0$ and $\mu = 1$, then the pairs of indices on the RHS of 38 are $(\alpha\beta, \mu\beta) = (00,10), (01,11), (02,12), (03,13)$. Thus

$$\delta^{\alpha}{}_{\mu} = \begin{cases} +1 & \text{if } \alpha = \mu \\ 0 & \text{otherwise} \end{cases} \tag{40}$$

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