DUAL TENSORS

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References: Charles W. Misner, Kip S. Thorne & John Archibald Wheeler, *Gravitation*, W.H. Freeman (1973). Exercise 3.14

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MTW define a *dual* with reference to vectors and rank-2 and rank-3 antisymmetric tensors. A dual is indicated by an asterisk before the tensor symbol, and the definitions are

$$*J_{\alpha\beta\gamma} \equiv J^{\mu}\varepsilon_{\mu\alpha\beta\gamma} \tag{1}$$

$$*F_{\alpha\beta} \equiv \frac{1}{2}F^{\mu\nu}\varepsilon_{\mu\nu\alpha\beta} \tag{2}$$

$$*B_{\alpha} \equiv \frac{1}{3!}B^{\lambda\mu\nu}\varepsilon_{\lambda\mu\nu\alpha} \tag{3}$$

Here, J^{μ} are the components of a vector J, $F^{\mu\nu}$ are the components of an antisymmetric rank-2 tensor, and $B^{\lambda\mu\nu}$ are the components of a rank-3 antisymmetric tensor.

Ex 3.14 (a) We are to show that taking the dual of a dual returns the original object, up to a sign. First, we look at the vector J, and we swap upper and lower indices in 1 so we can write

$$*J^{\alpha\beta\gamma} = J_{\mu}\varepsilon^{\mu\alpha\beta\gamma} \tag{4}$$

Taking the dual of this, we get, using 3

$$**J_{\delta} = \frac{1}{3!}J_{\mu}\varepsilon^{\mu\alpha\beta\gamma}\varepsilon_{\alpha\beta\gamma\delta} \tag{5}$$

$$= -\frac{1}{3!}J_{\mu}\varepsilon^{\alpha\beta\gamma\mu}\varepsilon_{\alpha\beta\gamma\delta} \tag{6}$$

where we swapped the μ index in $\varepsilon^{\mu\alpha\beta\gamma}$ 3 times to produce $\varepsilon^{\alpha\beta\gamma\mu}$. Since this is an odd number of swaps, the sign gets flipped.

We can now use the earlier result

$$\varepsilon^{\alpha\beta\gamma\mu}\varepsilon_{\alpha\beta\gamma\delta} = -6\delta^{\mu}{}_{\delta} \tag{7}$$

so we have

$$**J_{\delta}=-\frac{1}{3!}J_{\mu}\left(-6\delta^{\mu}{}_{\delta}\right)=J_{\delta} \tag{8}$$

Thus we have the result

$$**J=J \tag{9}$$

Now consider F. We have

$$*F^{\alpha\beta}=\frac{1}{2}F_{\mu\nu}\varepsilon^{\mu\nu\alpha\beta} \tag{10}$$

so the double-dual is, using 2

$$**F_{\gamma\delta}=\frac{1}{2}\left(\frac{1}{2}F_{\mu\nu}\varepsilon^{\mu\nu\alpha\beta}\right)\varepsilon_{\alpha\beta\gamma\delta} \tag{11}$$

$$=\frac{1}{4}F_{\mu\nu}\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\gamma\delta\alpha\beta} \tag{12}$$

where we performed an even number of swaps to map the indices in $\varepsilon_{\alpha\beta\gamma\delta}$ to $\varepsilon_{\gamma\delta\alpha\beta}$, so there is no change of sign. We now use the earlier result

$$\varepsilon^{\mu\nu\alpha\beta}\varepsilon_{\gamma\delta\alpha\beta}=-2\delta^{\mu\nu}{}_{\gamma\delta} \tag{13}$$

so we have

$$**F_{\gamma\delta}=-\frac{1}{2}F_{\mu\nu}\delta^{\mu\nu}{}_{\gamma\delta} \tag{14}$$

We know that

$$\delta^{\mu\nu}{}_{\gamma\delta}=\begin{cases}+1 & \text{if } \gamma\delta \text{ is an even permutation of } \mu\nu\\ -1 & \text{if } \gamma\delta \text{ is an odd permutation of } \mu\nu\\ 0 & \text{otherwise}\end{cases} \tag{15}$$

so doing the sum over $\mu\nu$ in 14 we have

$$**F_{\gamma\delta}=-\frac{1}{2}\left(F_{\gamma\delta}-F_{\delta\gamma}\right) \tag{16}$$

$$=-\frac{1}{2}\left(F_{\gamma\delta}+F_{\gamma\delta}\right) \tag{17}$$

$$=-F_{\gamma\delta} \tag{18}$$

where in the second line we used the condition that F is antisymmetric. Therefore

$$**F=-F \tag{19}$$

Finally, we look at the double dual of B. We have

$$*B^{\alpha}=\frac{1}{3!}B_{\lambda\mu\nu}\varepsilon^{\lambda\mu\nu\alpha} \tag{20}$$

so we have from 1

$$**B_{\beta\gamma\delta}=\frac{1}{3!}B_{\lambda\mu\nu}\varepsilon^{\lambda\mu\nu\alpha}\varepsilon_{\alpha\beta\gamma\delta} \tag{21}$$

$$=-\frac{1}{3!}B_{\lambda\mu\nu}\varepsilon^{\lambda\mu\nu\alpha}\varepsilon_{\beta\gamma\delta\alpha} \tag{22}$$

where we used to 3 swaps to convert $\varepsilon_{\alpha\beta\gamma\delta}$ to $-\varepsilon_{\beta\gamma\delta\alpha}$. Using the earlier result

$$\varepsilon^{\lambda\mu\nu\alpha}\varepsilon_{\beta\gamma\delta\alpha}=-\delta^{\lambda\mu\nu}{}_{\beta\gamma\delta} \tag{23}$$

we have

$$**B_{\beta\gamma\delta}=\frac{1}{3!}B_{\lambda\mu\nu}\delta^{\lambda\mu\nu}{}_{\beta\gamma\delta} \tag{24}$$

We know that

$$\delta^{\lambda\mu\nu}{}_{\beta\gamma\delta}=\begin{cases}+1 & \text{if } \lambda\mu\nu \text{ is an even permutation of } \beta\gamma\delta\\ -1 & \text{if } \lambda\mu\nu \text{ is an odd permutation of } \beta\gamma\delta\\ 0 & \text{otherwise}\end{cases} \tag{25}$$

so doing the sum over $\lambda\mu\nu$ in 24 gives us

$$**B_{\beta\gamma\delta}=\frac{1}{3!}\left(B_{\beta\gamma\delta}+B_{\gamma\delta\beta}+B_{\delta\beta\gamma}-B_{\gamma\beta\delta}-B_{\beta\delta\gamma}-B_{\delta\gamma\beta}\right) \tag{26}$$

$$=\frac{1}{3!}\times 6B_{\beta\gamma\delta} \tag{27}$$

$$=B_{\beta\gamma\delta} \tag{28}$$

where the second line follows because B is totally antisymmetric, so an even permutation of $\beta\gamma\delta$ gives $B_{\beta\gamma\delta}$ and an odd permutation gives $-B_{\beta\gamma\delta}$. Therefore

$$**B=B \tag{29}$$

(b) We can write out the components to verify these conclusions. From 1 we have, for example

$$\begin{aligned} *J_{012} &= J^3 \varepsilon_{3012} = -J^3 \\ *J_{123} &= J^0 \varepsilon_{0123} = J^0 \\ *J_{230} &= J^1 \varepsilon_{1230} = -J^1 \\ *J_{013} &= J^2 \varepsilon_{2013} = J^2 \end{aligned} \tag{30}$$

Permutations of the indices on the LHS can be found by swapping the indices in the ε and deducing the corresponding sign. Thus $*J$ and J contain the same information.

For F, we have from 2, for example (remember F is antisymmetric):

$$\begin{aligned} *F_{01} &= \frac{1}{2} F^{\mu\nu} \varepsilon_{\mu\nu 01} = \frac{1}{2} \left(F^{23} - F^{32} \right) = F^{23} \\ *F_{02} &= \frac{1}{2} F^{\mu\nu} \varepsilon_{\mu\nu 02} = \frac{1}{2} \left(-F^{13} + F^{31} \right) = -F^{13} \\ *F_{23} &= \frac{1}{2} F^{\mu\nu} \varepsilon_{\mu\nu 23} = \frac{1}{2} \left(F^{01} - F^{10} \right) = F^{01} \end{aligned} \tag{31}$$

and so on.

For B, we have from 3, for example

$$\begin{aligned} *B_0 &= \frac{1}{6} B^{\mu\nu\lambda} \varepsilon_{\mu\nu\lambda 0} \\ &= \frac{1}{6} \left(-B^{123} - B^{231} - B^{312} + B^{213} + B^{132} + B^{321} \right) \\ &= -B^{123} \end{aligned} \tag{32}$$

and so on.

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