GEOMETRIC VERSIONS OF MAXWELL'S EQUATIONS

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References: Charles W. Misner, Kip S. Thorne & John Archibald Wheeler, *Gravitation*, W.H. Freeman (1973). Exercise 3.15

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We've seen that Maxwell's equations can be written in terms of the electromagnetic field tensor $F_{\alpha\beta}$ in the compact form

$$F_{\alpha\beta,\gamma}+F_{\beta\gamma,\alpha}+F_{\gamma\alpha,\beta}=0 \tag{1}$$

with

$$F_{\alpha\beta}=\left[\begin{array}{cccc} 0 & -E_x & -E_y & -E_z\\ E_x & 0 & B_z & -B_y\\ E_y & -B_z & 0 & B_x\\ E_z & B_y & -B_x & 0 \end{array}\right] \tag{2}$$

and a comma before a subscript indicates a derivative with respect to the corresponding spacetime coordinate, so that

$$F_{\alpha\beta,\gamma}\equiv\frac{\partial F_{\alpha\beta}}{\partial x^{\gamma}} \tag{3}$$

The dual of an antisymmetric tensor such as $F_{\alpha\beta}$ is defined as

$$*F_{\alpha\beta}\equiv\frac{1}{2}F^{\mu\nu}\varepsilon_{\mu\nu\alpha\beta} \tag{4}$$

The divergence of a tensor is defined as the contraction of one of its indices with a derivative index, as in

$$\nabla\cdot F\equiv F^{\alpha\beta}{}_{,\alpha} \tag{5}$$

where there is an implied sum over α. Note that the divergence requires the index and corresponding derivative to be oppositely positioned, so that we must have (for example) the tensor index α as an upper index and the derivative index α as a lower index.

Ex 3.15. In the exercise we're asked to show that 1 can be written as the divergence of the dual $*F$, that is

$$\nabla\cdot *F=*F^{\alpha}{}_{\beta,\alpha}=0 \tag{6}$$

is equivalent to 1.

In order to show this, it's important that we get the indices in the correct locations. To calculate a divergence, we need the first index of $*F$ to be upper and the derivative index to be lower. That is, we want $*F^{\alpha}{}_{\beta,\alpha}$. There may be a quick way to do this for those more at ease with tensor indices, but I find it helpful to do all the steps. First, since 1 has all its indices as lower ones, we can rewrite 4 as (all repeated indices are summed):

$$*F_{\alpha\beta} \equiv \frac{1}{2}\eta^{\mu\rho}\eta^{\nu\sigma}F_{\rho\sigma}\varepsilon_{\mu\nu\alpha\beta} \tag{7}$$

with the usual metric tensor

$$\eta = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{8}$$

Equation 7 isn't as bad as it looks, since η is diagonal. We can work through it to determine the complete $*F_{\alpha\beta}$ matrix. For example, to find $*F_{01}$ we set $\alpha = 0$ and $\beta = 1$. This means that the only nonzero entries will be with $\mu = 2$ and $\nu = 3$ or $\nu = 2$ and $\mu = 3$, since these are the only values for which $\varepsilon_{\mu\nu\alpha\beta}$ is nonzero. If $\mu = 2$ and $\nu = 3$, the term in 7 is

$$\frac{1}{2}\eta^{2\rho}\eta^{3\sigma}F_{\rho\sigma}\varepsilon_{2301} \tag{9}$$

The value of ε_{2301} is +1, since an even number of swaps are required to convert the indices from 2301 to 0123. From 8, we see that the only nonzero elements of η are $\eta^{22} = \eta^{33} = 1$, so the term comes out to

$$\frac{1}{2}\eta^{2\rho}\eta^{3\sigma}F_{\rho\sigma}\varepsilon_{2301} = \frac{1}{2}F_{23} \tag{10}$$

Swapping $\mu \leftrightarrow \nu$ so that $\nu = 2$ and $\mu = 3$ causes ε_{2301} to become ε_{3201} which is -1, since it is a single swap from ε_{2301}. Thus

$$\frac{1}{2}\eta^{3\rho}\eta^{2\sigma}F_{\rho\sigma}\varepsilon_{3201} = -\frac{1}{2}F_{32} \tag{11}$$

From 2, $F_{\alpha\beta}$ is antisymmetric, so $-F_{32} = F_{23}$ and we have

$$*F_{01} = \frac{1}{2}\eta^{\mu\rho}\eta^{\nu\sigma}F_{\rho\sigma}\varepsilon_{\mu\nu01} \tag{12}$$

$$= \frac{1}{2}\left(F_{23} + F_{23}\right) \tag{13}$$

$$= F_{23} \tag{14}$$

This may seem like a lot of work to get just a single element, but if you work through a couple of them, you'll see that it's fairly straightforward to calculate the other elements which are all done the same way. We can also note that $*F_{\alpha\beta} = -*F_{\beta\alpha}$, since swapping $\alpha \leftrightarrow \beta$ in 7 swaps $\alpha \leftrightarrow \beta$ in $\varepsilon_{\mu\nu\alpha\beta}$ which changes the sign without doing anything else. The result is

$$*F_{\alpha\beta} = \begin{bmatrix} 0 & F_{23} & F_{31} & F_{12} \\ -F_{23} & 0 & F_{30} & F_{02} \\ -F_{31} & -F_{30} & 0 & F_{10} \\ -F_{12} & -F_{02} & -F_{10} & 0 \end{bmatrix} \tag{15}$$

However, to find $\nabla \cdot *F = *F^{\alpha}{}_{\beta,\alpha}$, we need the first index to be an upper index. We again use the metric to do this so we have

$$*F^{\alpha}{}_{\beta} = \eta^{\alpha\gamma} * F_{\gamma\beta} \tag{16}$$

Applying this to 15, we see that the only effect is to change the sign of the first row, so we have

$$*F^{\alpha}{}_{\beta} = \begin{bmatrix} 0 & -F_{23} & -F_{31} & -F_{12} \\ -F_{23} & 0 & F_{30} & F_{02} \\ -F_{31} & -F_{30} & 0 & F_{10} \\ -F_{12} & -F_{02} & -F_{10} & 0 \end{bmatrix} \tag{17}$$

Using the antisymmetry of F, we can write this without the minus signs as

$$*F^{\alpha}{}_{\beta} = \begin{bmatrix} 0 & F_{32} & F_{13} & F_{21} \\ F_{32} & 0 & F_{30} & F_{02} \\ F_{13} & F_{03} & 0 & F_{10} \\ F_{21} & F_{20} & F_{01} & 0 \end{bmatrix} \tag{18}$$

We can now find the divergence. This gives us 4 equations, one for each value of β. The results are

β	$*F^{\alpha}{}_{\beta,\alpha}$
0	$F_{32,1}+F_{13,2}+F_{21,3}$
1	$F_{32,0}+F_{03,2}+F_{20,3}$
2	$F_{13,0}+F_{30,1}+F_{01,3}$
3	$F_{21,0}+F_{02,1}+F_{10,2}$

All 4 of these rows are of the form 1 so we see that these Maxwell's equations are indeed the same as 6.

The other part of the question asks us to show that the second set of Maxwell's equations can be written as

$$\nabla \cdot F = 4\pi J \tag{19}$$

where J is the 4-current. If we take the divergence on the second slot of $F^{\alpha\beta}$ then we have

$$F^{\alpha\beta}{}_{,\beta} = 4\pi J^{\alpha} \tag{20}$$

As the result is already in the required form, I can't see that we need to do anything else.

The equations 6 and 19 are called *geometric* since they express Maxwell's equations in a coordinate-free form, using only the general form of tensors. Remember that any equation using components of tensors relies on a particular Lorentz frame.

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