CHARGE CONSERVATION

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: Charles W. Misner, Kip S. Thorne & John Archibald Wheeler, *Gravitation*, W.H. Freeman (1973). Exercise 3.16

Post date: 6 Aug 2020.

From the expression of Maxwell's equations given earlier

$$F^{\alpha\beta}{}_{,\beta} = 4\pi J^{\alpha} \tag{1}$$

we may derive the equation of conservation of charge. We take the divergence of 1 with respect to the index α to get

$$F^{\alpha\beta}{}_{,\beta,\alpha} = 4\pi J^{\alpha}{}_{,\alpha} \tag{2}$$

We can write the LHS as

$$F^{\alpha\beta}{}_{,\beta,\alpha} = F^{\beta\alpha}{}_{,\alpha,\beta} \tag{3}$$

$$= F^{\beta\alpha}{}_{,\beta,\alpha} \tag{4}$$

$$= -F^{\alpha\beta}{}_{,\beta,\alpha} \tag{5}$$

In the first line, we swapped the dummy labels $\alpha \leftrightarrow \beta$. In the second line, we reversed the order of derivatives, since the order doesn't matter. In the third line, we used the antisymmetry of $F^{\alpha\beta} = -F^{\beta\alpha}$. Thus

$$F^{\alpha\beta}{}_{,\beta,\alpha} = -F^{\alpha\beta}{}_{,\beta,\alpha} \tag{6}$$

so we must have

$$F^{\alpha\beta}{}_{,\beta,\alpha} = 4\pi J^{\alpha}{}_{,\alpha} = 0 \tag{7}$$

Since the four-current J is composed of the 3-current vector $\boldsymbol{J}$ and the charge density $J^0 = \rho$, the RHS of this last equation can be written as

$$J^{\alpha}{}_{,\alpha} = \nabla \cdot \boldsymbol{J} + \frac{\partial J^0}{\partial t} = 0 \tag{8}$$

Integrating over all space and using Gauss's theorem to convert the volume integral of a divergence to a surface integral, we have

$$\int_V \nabla\cdot\boldsymbol{J}\,d^3x = -\frac{\partial}{\partial t}\int_V J^0 d^3x \tag{9}$$

$$\int_A \boldsymbol{J}\cdot\hat{\boldsymbol{n}}d\boldsymbol{a} = -\frac{\partial Q}{\partial t} \tag{10}$$

This states that the integral of the current over a surface A bounding the volume V is the negative of the rate of change of charge Q within V. If we take the surface far enough out so that no current crosses it, then the surface integral is zero, and the total charge remains unchanged.

PINGBACKS

Pingback: Gauss's law for four-current