ELECTROMAGNETIC VECTOR POTENTIAL AND GAUGES

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: Charles W. Misner, Kip S. Thorne & John Archibald Wheeler, *Gravitation*, W.H. Freeman (1973). Exercise 3.17.

Post date: 6 Aug 2020.

The electromagnetic vector potential A is defined so that the electromagnetic field tensor F satisfies

$$F = -(\text{antisymmetric part of } \nabla A) \tag{1}$$

This is the geometric form of the definition, which can be written in terms of components as

$$F_{\alpha\beta} = A_{\beta,\alpha} - A_{\alpha,\beta} \tag{2}$$

Ex 3.17(a). We can use this definition to relate A to the electric and magnetic fields. The explicit form of $F_{\mu\nu}$ is

$$F_{\alpha\beta} = \begin{bmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{bmatrix} \tag{3}$$

Take, for example, B_x:

$$B_x = F_{23} \tag{4}$$

$$= A_{3,2} - A_{2,3} \tag{5}$$

$$= \frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z} \tag{6}$$

$$= (\nabla \times \boldsymbol{A})_x \tag{7}$$

We can work out similar expressions for B_y and B_z so we see that

$$\boldsymbol{B} = \nabla \times \boldsymbol{A} \tag{8}$$

Now consider E_x:

$$
\begin{aligned}
E_x &= F_{10} &(9)\\
&= A_{0,1} - A_{1,0} &(10)\\
&= -A^0{}_{,1} - A^1{}_{,0} &(11)\\
&= -\frac{\partial A^0}{\partial x} - \frac{\partial A^1}{\partial t} &(12)\\
&= -\left(\nabla A^0\right)_x - \left(\frac{\partial \boldsymbol{A}}{\partial t}\right)_x &(13)
\end{aligned}
$$

In the third line, we used the fact that raising the 0 index changes the sign ($A^0 = -A_0$) while raising indices 1, 2 or 3 doesn't change the sign. We can work out the other 2 components similarly, so we get

$$\boldsymbol{E} = -\frac{\partial \boldsymbol{A}}{\partial t} - \nabla A^0 \tag{14}$$

(b) One of Maxwell's equations is

$$F^{\alpha\beta}{}_{,\beta} = 4\pi J^\alpha \tag{15}$$

Applying this to 2 we have

$$F^{\alpha\beta}{}_{,\beta} = A^{\beta,\alpha}{}_{\beta} - A^{\alpha,\beta}{}_{,\beta} = 4\pi J^\alpha \tag{16}$$

Since the order of derivatives doesn't matter, we can write this as (relabelling $\beta \to \mu$ to use MTW's notation):

$$A^{\alpha,\mu}{}_{,\mu} - A^{\mu}{}_{,\mu}{}^{,\alpha} = -4\pi J^\alpha \tag{17}$$

(c) To introduce the idea of different gauges, suppose we have an arbitrary function ϕ. We can define a new vector potential A' by adding the gradient of ϕ to the old vector potential A:

$$A' = A + \boldsymbol{d}\phi \tag{18}$$

where

$$(\boldsymbol{d}\phi)_\nu = \phi_{,\nu} \tag{19}$$

From 2, we can write F in terms of A':

$$A'_{\beta,\alpha} - A'_{\alpha,\beta} = A_{\beta,\alpha} + \phi_{\alpha,\beta} - A_{\alpha,\beta} - \phi_{,\beta,\alpha} \tag{20}$$

Since the order of derivatives doesn't matter, $\phi_{,\alpha,\beta} = \phi_{,\beta,\alpha}$ and the terms in ϕ cancel out. Thus using 18 instead of A leaves F unchanged.

(d) We've looked at the Coulomb and Lorenz gauges before but this was in terms of ordinary 3-d vectors, so it's worth reworking it here. One point first, however.

MTW make the common mistake of calling the Lorenz (after Ludvig Lorenz, Danish physicist, 1829-1891) gauge the 'Lorentz' (after Hendrik Lorentz, Dutch physicist, 1853 - 1928; after whom the Lorentz transformations are named) gauge.

A gauge is defined by choosing ϕ in 18 so that a certain condition is satisfied. In the Lorenz gauge, we require

$$\nabla\cdot A=0 \tag{21}$$

Note that this is a four-vector equation, so be careful not to confuse it with the 3-vector condition $\nabla\cdot\boldsymbol{A}=0$, which defines the Coulomb gauge.

The condition 21 is equivalent to

$$\nabla\cdot\boldsymbol{A}'=-\frac{\partial A'^0}{\partial t} \tag{22}$$

From 18 we have

$$A^{\mu}{}_{,\mu}+\phi^{,\mu}{}_{,\mu}=0 \tag{23}$$

Thus the Lorenz gauge can be satisfied if we can find ϕ such that

$$\phi^{,\mu}{}_{,\mu}=-A^{\mu}{}_{,\mu} \tag{24}$$

In practice, this is a second order partial differential equation which can be difficult (or impossible) to solve exactly, but for some examples see the earlier post.

Once we have found a Lorenz gauge, the condition 21 can be written in component form as

$$\nabla\cdot A=A^{\mu}{}_{,\mu}=0 \tag{25}$$

Thus using this gauge, we see that the second term on the LHS of 17 is zero, and Maxwell's equations become

$$A^{\alpha,\mu}{}_{,\mu}=-4\pi J^{\alpha} \tag{26}$$

This is written in geometric form as

$$\Box A=-4\pi J \tag{27}$$

where

$$\Box A\equiv A^{\alpha,\mu}{}_{,\mu}\boldsymbol{e}_{\alpha} \tag{28}$$

where $\boldsymbol{e}_\alpha$ are the unit vectors.