DIVERGENCE OF ELECTROMAGNETIC STRESS-ENERGY TENSOR

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References: Charles W. Misner, Kip S. Thorne & John Archibald Wheeler, *Gravitation*, W.H. Freeman (1973). Exercise 3.18.

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The electromagnetic stress-energy tensor is defined by MTW in terms of the electromagnetic field tensor as

$$T^{\mu\nu}=\frac{1}{4\pi}\left(F^{\mu\alpha}F^{\nu}{}_{\alpha}-\frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right) \tag{1}$$

Ex 3.18 (a) The divergence on the second slot of T is therefore

$$\begin{aligned}T^{\mu\nu}{}_{,\nu}&=\frac{1}{4\pi}\left[F^{\mu\alpha}{}_{,\nu}F^{\nu}{}_{\alpha}+F^{\mu\alpha}F^{\nu}{}_{\alpha,\nu}-\right.\\&\qquad\left.\frac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}{}_{,\nu}F_{\alpha\beta}-\frac{1}{4}\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta,\nu}\right] \qquad (2)\\&=\frac{1}{4\pi}\left[F^{\mu\alpha}{}_{,\nu}F^{\nu}{}_{\alpha}+F^{\mu\alpha}F^{\nu}{}_{\alpha,\nu}-\right.\\&\qquad\left.\frac{1}{4}F^{\alpha\beta,\mu}F_{\alpha\beta}-\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}{}^{,\mu}\right] \qquad (3)\end{aligned}$$

We can write the first term in the last line as

$$\begin{aligned}\frac{1}{4}F^{\alpha\beta,\mu}F_{\alpha\beta}&=\frac{1}{4}\eta^{\alpha\gamma}\eta^{\beta\delta}F_{\gamma\delta}{}^{,\mu} \qquad (4)\\&=\frac{1}{4}F_{\gamma\delta}{}^{,\mu}F^{\gamma\delta} \qquad (5)\\&=\frac{1}{4}F^{\alpha\beta}F_{\alpha\beta}{}^{,\mu} \qquad (6)\end{aligned}$$

where in the last line we've renamed the dummy indices as $\gamma\to\alpha$ and $\delta\to\beta$. Putting this back into 3 we have

$$T^{\mu\nu}{}_{,\nu}=\frac{1}{4\pi}\left[F^{\mu\alpha}{}_{,\nu}F^{\nu}{}_{\alpha}+F^{\mu\alpha}F^{\nu}{}_{\alpha,\nu}-\frac{1}{2}F^{\alpha\beta}F_{\alpha\beta}{}^{,\mu}\right] \tag{7}$$

(b) We now wish to lower the μ index in $T^{\mu\nu}{}_{,\nu}$. We have

$$4\pi T_{\mu}{}^{\nu}{}_{,\nu} = 4\pi\eta_{\mu\gamma}T^{\gamma\nu}{}_{,\nu} \tag{8}$$

$$= F_{\mu}{}^{\alpha}{}_{,\nu}F^{\nu}{}_{\alpha} + F_{\mu}{}^{\alpha}F^{\nu}{}_{\alpha,\nu} - \frac{1}{2}F^{\alpha\beta}F_{\alpha\beta,\mu} \tag{9}$$

$$= F_{\mu\alpha,\nu}F^{\nu\alpha} + F_{\mu\alpha}F^{\nu\alpha}{}_{,\nu} - \frac{1}{2}F^{\alpha\beta}F_{\alpha\beta,\mu} \tag{10}$$

To get the last line, we've raised and lowered the α index in the first two terms. If you don't believe this can be done, you can write it out longhand by using the metrics $\eta^{\mu\nu}$ and $\eta_{\mu\nu}$ as we did in 4.

We now use the antisymmetry of $F^{\alpha\beta}$ to write the second term of 10 as

$$F_{\mu\alpha}F^{\nu\alpha}{}_{,\nu} = -F_{\mu\alpha}F^{\alpha\nu}{}_{,\nu} \tag{11}$$

By swapping indices and using asymmetry, we can write the first term of 10 in two equivalent ways:

$$F_{\mu\alpha,\nu}F^{\nu\alpha} = -F_{\mu\alpha,\nu}F^{\alpha\nu} \tag{12}$$

$$= -F_{\alpha\mu,\nu}F^{\nu\alpha} \tag{13}$$

Therefore, we have

$$F_{\mu\alpha,\nu}F^{\nu\alpha} = -\frac{1}{2}F_{\mu\alpha,\nu}F^{\alpha\nu} - \frac{1}{2}F_{\alpha\mu,\nu}F^{\nu\alpha} \tag{14}$$

We can rename $\nu\to\beta$ in the first term, and $\alpha\to\beta$ and $\nu\to\alpha$ in the second term to get

$$F_{\mu\alpha,\nu}F^{\nu\alpha} - \frac{1}{2}F_{\mu\alpha,\beta}F^{\alpha\beta} - \frac{1}{2}F_{\beta\mu,\alpha}F^{\alpha\beta} \tag{15}$$

Inserting this into 10 we have

$$T_{\mu}{}^{\nu}{}_{,\nu} = \frac{1}{4\pi}\left[-F_{\mu\alpha}F^{\alpha\nu}{}_{,\nu} - \frac{1}{2}F^{\alpha\beta}\left(F_{\alpha\beta,\mu} + F_{\mu\alpha,\beta} + F_{\beta\mu,\alpha}\right)\right] \tag{16}$$

(c) We can now use Maxwell's equations in the form

$$F_{\alpha\beta,\mu} + F_{\mu\alpha,\beta} + F_{\beta\mu,\alpha} = 0 \tag{17}$$

to write 16 as

$$T_{\mu}{}^{\nu}{}_{,\nu} = -\frac{1}{4\pi}F_{\mu\alpha}F^{\alpha\nu}{}_{,\nu} \tag{18}$$

Using the other set of Maxwell's equations

$$F^{\alpha\nu}{}_{,\nu} = 4\pi J^{\alpha} \tag{19}$$

we have

$$T_{\mu}{}^{\nu}{}_{,\nu} = -\frac{1}{4\pi} F_{\mu\alpha} 4\pi J^{\alpha} \tag{20}$$

$$= -F_{\mu\alpha} J^{\alpha} \tag{21}$$

Raising and lowering indices gives

$$T^{\mu\nu}{}_{,\nu} = -F^{\mu\alpha} J_{\alpha} \tag{22}$$