ELECTROMAGNETIC STRESS-ENERGY TENSOR; TENSION & PRESSURE

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References: Charles W. Misner, Kip S. Thorne & John Archibald Wheeler, *Gravitation*, W.H. Freeman (1973). Exercise 5.1.

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At the start of Chapter 5, MTW define the stress-energy tensor and give a couple of examples. We've already seen the tensor for a perfect fluid and the electromagnetic stress-energy tensor. In MTW's notation, the tensor is

$$4\pi T^{\mu\nu}=F^{\mu\alpha}F^{\nu}{}_{\alpha}-\frac{1}{4}\eta^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} \tag{1}$$

where the electromagnetic field tensor is

$$\left\|F^{\alpha}{}_{\beta}\right\|=\begin{bmatrix}0 & E_1 & E_2 & E_3\\ E_1 & 0 & B_3 & -B_2\\ E_2 & -B_3 & 0 & B_1\\ E_3 & B_2 & -B_1 & 0\end{bmatrix} \tag{2}$$

or, with both indices lowered

$$\left\|F_{\alpha\beta}\right\|=\begin{bmatrix}0 & -E_1 & -E_2 & -E_3\\ E_1 & 0 & B_3 & -B_2\\ E_2 & -B_3 & 0 & B_1\\ E_3 & B_2 & -B_1 & 0\end{bmatrix} \tag{3}$$

and with both indices raised

$$\left\|F^{\alpha\beta}\right\|=\begin{bmatrix}0 & E_1 & E_2 & E_3\\ -E_1 & 0 & B_3 & -B_2\\ -E_2 & -B_3 & 0 & B_1\\ -E_3 & B_2 & -B_1 & 0\end{bmatrix} \tag{4}$$

and the metric tensor is

$$\eta=\begin{bmatrix}-1 & 0 & 0 & 0\\ 0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\end{bmatrix} \tag{5}$$

By working out the components of 1 we can find its expression in terms of $\boldsymbol{E}$ and $\boldsymbol{B}$. We have

$$4\pi T^{00} = F^{0\alpha}F^{0}{}_{\alpha} + \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} \tag{6}$$

$$= \boldsymbol{E}^2 + \frac{1}{4}\left[-2\boldsymbol{E}^2 + 2\boldsymbol{B}^2\right] \tag{7}$$

$$= \frac{1}{2}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right) \tag{8}$$

$$T^{00} = \frac{1}{8\pi}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right) \tag{9}$$

For the other top row entries, we can work out an example:

$$4\pi T^{01} = F^{0\alpha}F^{1}{}_{\alpha} \tag{10}$$

$$= E_2B_3 - E_3B_2 \tag{11}$$

$$= (\boldsymbol{E}\times\boldsymbol{B})^1 \tag{12}$$

The other two components work out the same way so we have

$$T^{0j} = T^{j0} = \frac{1}{4\pi}(\boldsymbol{E}\times\boldsymbol{B})^j \tag{13}$$

For the purely spatial components, we again work out a couple of examples. First, a diagonal entry:

$$4\pi T^{11} = F^{1\alpha}F^{1}{}_{\alpha} - \frac{1}{4}F_{\alpha\beta}F^{\alpha\beta} \tag{14}$$

$$= -E_1^2 + B_3^2 + B_2^2 - \frac{1}{4}\left[-2\boldsymbol{E}^2 + 2\boldsymbol{B}^2\right] \tag{15}$$

$$= -E_1^2 - B_1^2 + \boldsymbol{B}^2 + \frac{1}{2}\boldsymbol{E}^2 - \frac{1}{2}\boldsymbol{B}^2 \tag{16}$$

$$= -\left(E_1^2 + B_1^2\right) + \frac{1}{2}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right) \tag{17}$$

Then, an off-diagonal entry:

$$4\pi T^{12} = F^{1\alpha}F^{2}{}_{\alpha} \tag{18}$$

$$= -E_1E_2 - B_1B_2 \tag{19}$$

The other components work out the same way, so we have

$$T^{00} = \frac{1}{8\pi}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right) \tag{20}$$

$$T^{0j} = T^{j0} = \frac{1}{4\pi}(\boldsymbol{E}\times\boldsymbol{B})^j \tag{21}$$

$$T^{jk} = T^{kj} = \frac{1}{4\pi}\left[-\left(E^jE^k + B^jB^k\right) + \frac{1}{2}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right)\delta^{kj}\right] \tag{22}$$

The final part of the problem asks us to verify that the stress tensor describes a tension $\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right)/8\pi$ along field lines and a pressure of $\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right)/8\pi$ perpendicular to the field lines. I was unable to find a clear description of exactly what is meant by 'tension' and 'pressure' in this context, so if anyone knows of a good description please let me know in a comment. The best I could come up with is something along these lines.

First, the stress tensor (as opposed to the stress-energy tensor) consists of the purely spatial components of T, so we have

$$T^{jk} = \frac{1}{4\pi}\begin{bmatrix} -E_1^2 - B_1^2 + \frac{1}{2}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right) & -E_1E_2 - B_1B_2 & -E_1E_3 - B_1B_3 \\ -E_1E_2 - B_1B_2 & -E_2^2 - B_2^2 + \frac{1}{2}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right) & -E_2E_3 - B_2B_3 \\ -E_1E_3 - B_1B_3 & -E_2E_3 - B_2B_3 & -E_3^2 - B_3^2 + \frac{1}{2}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right) \end{bmatrix} \tag{23}$$

Consider the simpler case where the electric field is zero, so we have only a magnetic field $\boldsymbol{B}$. Choose a unit vector $\boldsymbol{n}$ parallel to $\boldsymbol{B}$ and another unit vector $\boldsymbol{u}$ perpendicular to $\boldsymbol{B}$. We then have

$$T^{jk} = \frac{1}{4\pi}\left(-B^jB^k + \frac{1}{2}\left(\boldsymbol{E}^2 + \boldsymbol{B}^2\right)\delta^{kj}\right) \tag{24}$$

or, in full:

$$T^{jk} = \frac{1}{4\pi}\begin{bmatrix} -B_1^2 + \frac{1}{2}\boldsymbol{B}^2 & -B_1B_2 & -B_1B_3 \\ -B_1B_2 & -B_2^2 + \frac{1}{2}\boldsymbol{B}^2 & -B_2B_3 \\ -B_1B_3 & -B_2B_3 & -B_3^2 + \frac{1}{2}\boldsymbol{B}^2 \end{bmatrix} \tag{25}$$

Now, (citing MTW eqn 5.13d) the diagonal components T^{jk} represent the components of momentum transfer (that is, force) of the j component of momentum in the k direction. If we project T^{jk} along a unit vector, we find the component of force in that direction. That is, along $\boldsymbol{n}$ (parallel to the lines of force)

$$T^{jk}n_k = \frac{1}{4\pi}\left(-B^jB^kn_k + \frac{1}{2}\boldsymbol{B}^2\delta^{kj}n_k\right) \tag{26}$$

$$= \frac{1}{4\pi}\left(-B^j|\boldsymbol{B}| + \frac{1}{2}\boldsymbol{B}^2n^j\right) \tag{27}$$

Since $\boldsymbol{B}$ is parallel to $\boldsymbol{n}$, then (assuming we've aligned things so that $\boldsymbol{B}$ and $\boldsymbol{n}$ both point along one of the coordinate axes, let's say along the $j=1$ axis), only one of these components is non-zero, that is, along the $j=1$ axis, so that $\boldsymbol{n}=(1,0,0)$ and $\boldsymbol{B}=(|\boldsymbol{B}|,0,0)$. In that case, $B^1=|\boldsymbol{B}|$ and we have

$$T^{1k}n_k = \frac{1}{4\pi}\left(-\boldsymbol{B}^2 + \frac{1}{2}\boldsymbol{B}^2\right) \tag{28}$$

$$= -\frac{\boldsymbol{B}^2}{8\pi} \tag{29}$$

The fact that the force is negative indicates that it's a force of tension.

Now if we consider the unit vector $\boldsymbol{u}=(0,1,0)$ say (we just require that $\boldsymbol{u}$ is perpendicular to $\boldsymbol{n}$) then

$$T^{2k}u_k = \frac{1}{4\pi}\left(-B^2B^ku_k + \frac{1}{2}\boldsymbol{B}^2\delta^{k2}u_k\right) \tag{30}$$

$$= \frac{1}{4\pi}\left(0 + \frac{1}{2}\boldsymbol{B}^2\right) \tag{31}$$

$$= \frac{\boldsymbol{B}^2}{8\pi} \tag{32}$$

where the second line follows since $\boldsymbol{B}\perp\boldsymbol{u}$ so $B^ku_k=\boldsymbol{B}\cdot\boldsymbol{u}=0$ (also $B^2=0$ (component 2 of $\boldsymbol{B}$, not B squared) since $\boldsymbol{B}=(|\boldsymbol{B}|,0,0)$) . In this case, the force is positive, which indicates a pressure in the direction perpendicular to the lines of force.

We could apply the same argument to the case where $\boldsymbol{B}=0$ and we have only an electric field, but I'm not sure how the argument would generalize to arbitrary electric and magnetic fields. As always, comments welcome.

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