GAUSS'S LAW FOR FOUR-CURRENT

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References: Charles W. Misner, Kip S. Thorne & John Archibald Wheeler, *Gravitation*, W.H. Freeman (1973). Exercise 5.2.

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We've seen that the electromagnetic four-current J has a zero four-divergence, which is expressed in geometric, or coordinate-free, form as

$$\nabla\cdot\mathsf{J}=0 \tag{1}$$

and in component form as

$$J^{\mu}{}_{,\mu}=0 \tag{2}$$

MTW discuss Gauss's theorem for relating surface and volume integrals in their Box 5.3. The integral form of 1 can be written as

$$\int_{\mathscr{V}}\nabla\cdot\mathsf{J}\,dt\,dx\,dy\,dz=0 \tag{3}$$

where $\mathscr{V}$ is a given volume in spacetime. In components, this is

$$\int_{\mathscr{V}}J^{\mu}{}_{,\mu}\,dt\,dx\,dy\,dz=0$$

Gauss's theorem allows us to transform this integral over a volume into an integral over the surface bounding the volume, so that

$$\int_{\partial\mathscr{V}}J^{\mu}\,d^{3}\Sigma_{\mu}=0 \tag{4}$$

where $d^3\Sigma_\mu$ represents a collection of surfaces that, when joined together, bound the volume $\mathscr{V}$. These surfaces are 3-dimensional, since $\mathscr{V}$ itself is a four-dimensional volume in spacetime. A surface may consist entirely of spatial dimensions, or it may contain a mixture of 2 space dimensions with 1 time dimension.

MTW give examples of this integral in the case of momentum conservation, using the electromagnetic stress-energy tensor $T^{\mu\nu}$. Here, we're asked to run through these examples for the case of the four-current J^{μ}.

Case (b). In this case, the surface consists of two slices through the same Lorentz frame, where each slice represents a constant time. In other words, the surface encloses all space between two fixed times in a given frame. The surfaces are joined at infinity by timelike surfaces, and it is assumed that all components of J^μ tend to zero at infinity fast enough so that their integrals over these joining timelike surfaces can be neglected. Remember that a 'spacelike surface' is one in which any two points, or events, cannot be joined by a light signal, so that there is a non-zero spatial distance between these two points in any Lorentz frame. The temporal order of these two events depends on the Lorentz frame, so that some frames will say that one event occurs before the other, while other frames will say the reverse. There is one frame in which the two events are judged to be simultaneous.

In the four-current case, we define surface $\mathscr{S}_1$ to be at the earlier time and orient its outward normal to point towards the past. The surface $\mathscr{S}_2$ is at the later time and its outward normal points towards the future. Thus in 4 we have $\partial\mathscr{V} = \mathscr{S}_2 - \mathscr{S}_1$. Since the time on each surface is a constant, the integral extends over all spatial coordinates. The sum over μ in 4 becomes a sum over the two times that are represented by the two surfaces, so it is the $\mu = 0$ component that appears in the integral. That is,

$$\int_{\partial\mathscr{V}} J^\mu \, d^3\Sigma_\mu = \int_{\mathscr{S}_2} J^0 \, dx \, dy \, dz - \int_{\mathscr{S}_1} J^0 \, dx \, dy \, dz = 0 \tag{5}$$

or

$$\int_{\mathscr{S}_2} J^0 \, dx \, dy \, dz = \int_{\mathscr{S}_1} J^0 \, dx \, dy \, dz \tag{6}$$

Since J^0 represents charge density, this is a statement that charge is conserved over time.

Case (c). In this case, surface $\mathscr{S}$ represents a fixed time in one Lorentz frame and $\bar{\mathscr{S}}$ a fixed time in a different Lorentz frame (see MTW Fig. 5.3(c)). As before, we imagine these two surfaces are joined by timelike surfaces at infinity, which we can ignore. In this case, it doesn't really matter which surface is defined to point towards the past and which towards the future, as long as the two surfaces point in opposite directions.

In this case, we can view the integral in the geometric form, where it says

$$\int_{\mathscr{S}} \mathsf{J} \cdot d^3\mathbf{\Sigma} = \int_{\bar{\mathscr{S}}} \mathsf{J} \cdot d^3\mathbf{\Sigma} \tag{7}$$

In $\mathscr{S}$, the integral could be written as

$$\int_{\mathscr{S}} \mathsf{J} \cdot d^3\mathbf{\Sigma} = \int_{\mathscr{S}} J^0 \, dx \, dy \, dz \tag{8}$$

if we use the coordinates in the $\mathscr{S}$ Lorentz frame, since in that frame, $\mathscr{S}$ is a surface with a fixed time. The surface $\bar{\mathscr{S}}$, however, does not have a fixed time in the $\mathscr{S}$ frame, so the integral on the RHS of 7 doesn't have such a simple form if we try to calculate it using $\mathscr{S}$ coordinates. If we calculate it using $\bar{\mathscr{S}}$ coordinates, however, it again reduces to a calculation of the total charge in all space, so 7 states that the total charge in all space (which is a scalar quantity) is the same in the two Lorentz frames.

Case (d). Here, the two surfaces $\mathscr{S}_A$ and $\mathscr{S}_B$ are again spacelike, but they don't necessarily represent constant times (see MTW Fig. 5.3(d)). However, they are still joined by timelike surfaces at infinity that we can ignore. We now have

$$\int_{\mathscr{S}_A} \mathsf{J}\cdot d^3\mathbf{\Sigma}=\int_{\mathscr{S}_B} \mathsf{J}\cdot d^3\mathbf{\Sigma} \tag{9}$$

which says that the total charge measured on the two surfaces must be equal.

Case (e). We now have two spacelike surfaces $\mathscr{S}$ and $\bar{\mathscr{S}}$ that are joined by two timelike surfaces on either side, but *not* at (spatial) infinity (see MTW Fig. 5.3(e)). In this case, the spatial volumes of $\mathscr{S}$ and $\bar{\mathscr{S}}$ do *not* include all of space, so the total charge on $\bar{\mathscr{S}}$ is equal to the total charge on $\mathscr{S}$ plus or minus any charge that flows in or out of the two timelike surfaces between $\mathscr{S}$ and $\bar{\mathscr{S}}$.

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