GAUSS'S LAW FOR ELECTRON-POSITRON CREATION IN STARS

Link to: physicspages home page.

To leave a comment or report an error, please use the auxiliary blog.

References: Charles W. Misner, Kip S. Thorne & John Archibald Wheeler, *Gravitation*, W.H. Freeman (1973). Exercise 5.3.

Post date: 22 Aug 2020.

In some stars, the temperature is high enough that there is enough energy to create electron-positron pairs, which can collide later to annihilate themselves. We can describe the number flux of these particles by the tensor S. This is analogous to the electromagnetic four-current J that we looked at earlier. In the four-current example, total charge is conserved and the four-current divergence is zero:

$$\nabla\cdot\mathsf{J}=0 \tag{1}$$

In the pair creation case, the total number of particles is not conserved, so the divergence is non-zero, and we have

$$\epsilon\equiv\nabla\cdot\mathsf{S} \tag{2}$$

The components of $\boldsymbol{S}$, by analogy with $\boldsymbol{J}$ are: S^0 is the number density of particles, and the spatial components S^j represent the flux of particles in each of the three spatial directions.

Using Gauss's theorem, we can relate the volume integral to a surface integral, so we have

$$\int_{\mathscr{V}}\nabla\cdot\mathsf{S}\,d^4\Omega=\oint_{\partial\mathscr{V}}\mathsf{S}\cdot d^3\boldsymbol{\Sigma} \tag{3}$$

where $\mathscr{V}$ is a spacetime volume enclosed by surface $\partial\mathscr{V}$, and $d^4\Omega$ is a four-volume element and $d^3\boldsymbol{\Sigma}$ is an outward-facing 3-surface element.

To make this concrete, we can consider case (b) from our earlier treatment of the electromagnetic four-current. Here, we have a surface bounded by two constant times t_1 and t_2 (as viewed in one Lorentz frame) and a spatial volume that covers all space. Thus we're considering how the number of particles in all space changes between two fixed times. In this case, the surface $\partial\mathscr{V}$ can be divided into two surfaces: a surface S_1 at $t=t_1$ and a second surface S_2 at $t=t_2$. These two surfaces are assumed to be connected by two timelike surfaces at infinity, and we further assume that $\boldsymbol{S}$ goes to zero fast enough at infinity that any integrals over these surfaces can be neglected. This is certainly true if the volume we're considering is that of a star.

In this case, we can write 3 in component form as

$$\oint_{\partial\mathscr{V}} \mathsf{S}\cdot d^3\mathbf{\Sigma} = \int_{S_2} S^0\, dx\, dy\, dz - \int_{S_1} S^0\, dx\, dy\, dz \tag{4}$$

$$= \int_{\mathscr{V}} \nabla\cdot\mathsf{S}\, d^4\Omega \tag{5}$$

$$= \int_{\mathscr{V}} \epsilon\, d^4\Omega \tag{6}$$

The expression 4 is the difference in particle number between times t_1 and t_2, and is equal, via 2, to the volume integral of ϵ in 6. This shows that ϵ is the rate of particle creation per unit volume and unit time.

The argument can be extended to other spacetime volumes in the same way as we did for the four-current, so that 3 can be written as

$$\int_{\mathscr{V}} \nabla\cdot\mathsf{S}\, d^4\Omega = \oint_{\partial\mathscr{V}} \mathsf{S}\cdot d^3\mathbf{\Sigma} = \int_{\mathscr{V}} \epsilon\, d^4\Omega \tag{7}$$

In other words, the total number of particles created (minus those annihilated) in a spacetime volume $\mathscr{V}$, represented by $\int_{\mathscr{V}} \epsilon\, d^4\Omega$, is equal to the four-flux S integrated over the surface $\partial\mathscr{V}$ bounding that spacetime volume.