INERTIAL MASS PER UNIT VOLUME

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References: Charles W. Misner, Kip S. Thorne & John Archibald Wheeler, *Gravitation*, W.H. Freeman (1973). Exercise 5.4.

Post date: 24 Aug 2020.

Here we look at a stressed medium that moves with a small velocity $|\boldsymbol{v}| \ll 1$ relative to a specific Lorentz (lab) frame. In the medium's rest frame, its stress-energy tensor is given by barred indices as $T^{\bar{\alpha}\bar{\beta}}$ and in the lab frame by unbarred indices $T^{\mu\nu}$.

Ex. 5.4 (a) The transformation of a tensor between the two frames is given by a Lorentz transformation, so we have

$$T^{\mu\nu} = \Lambda^{\mu}{}_{\bar{\alpha}}\Lambda^{\nu}{}_{\bar{\beta}}T^{\bar{\alpha}\bar{\beta}} \tag{1}$$

It's easiest if we consider motion to be along one of the coordinate axes, say the x axis, which we can take to correspond to the $\bar{x}$ axis. In that case, the Lorentz transformation is

$$\left\|\Lambda^{\mu}{}_{\bar{\alpha}}\right\| = \begin{bmatrix} \gamma & v\gamma & 0 & 0 \\ v\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

Writing out the transformation for T^{01}, for example, we have

$$T^{01} = \Lambda^{0}{}_{\bar{\alpha}}\Lambda^{1}{}_{\bar{\beta}}T^{\bar{\alpha}\bar{\beta}} \tag{3}$$

$$= \Lambda^{1}{}_{\bar{\beta}}\left(\gamma T^{\bar{0}\bar{\beta}} + v\gamma T^{\bar{1}\bar{\beta}}\right) \tag{4}$$

$$= v\gamma\left(\gamma T^{\bar{0}\bar{0}} + v\gamma T^{\bar{1}\bar{0}}\right) + \gamma\left(\gamma T^{\bar{0}\bar{1}} + v\gamma T^{\bar{1}\bar{1}}\right) \tag{5}$$

For $v \ll 1$, we can use the approximation

$$\gamma = \frac{1}{\sqrt{1-v^2}} \approx 1 \tag{6}$$

and neglect all terms of order $\mathcal{O}\left(v^2\right)$. This gives

$$T^{01} = vT^{\bar{0}\bar{0}} + T^{\bar{0}\bar{1}} + vT^{\bar{1}\bar{1}} \tag{7}$$

In the medium's rest frame, there is no energy flux so $T^{\bar{0}\bar{1}}=0$ and we're left with

$$T^{01}=vT^{\bar{0}\bar{0}}+vT^{\bar{1}\bar{1}} \tag{8}$$

If the velocity $\boldsymbol{v}$ is along either of the other two coordinate axes y or z, we'd get the same result with '1' replaced by '2' or '3'.

Presumably, we can generalize this to the case where $\boldsymbol{v}$ is along some arbitrary direction, but I'll leave it there as the idea is clear enough. This gives us MTW's equations 5.51 and 5.52 valid for motion along one of the coordinate axes:

$$T^{jk}=\sum_k m^{jk}v^k \tag{9}$$

with

$$m^{jk}\equiv T^{\bar{0}\bar{0}}\delta^{jk}+T^{\bar{j}\bar{k}} \tag{10}$$

(b) In Newtonian physics, the transport of mass-energy arises from the motion of rest mass and from the work done by a component of a force in a given direction. The components of the stress-energy tensor have the interpretations:

- T^{00} is the mass-energy density;
- T^{jk} is the force on a volume element at position $x^j+\epsilon$ in the k direction due to matter at position $x^j-\epsilon$. That is, it's the transfer of momentum component k in the j direction (or vice versa, since $T^{jk}=T^{kj}$ as shown in MTW's §5.7).

In the lab frame, the flux in the j direction arises from the motion of the rest mass due to v^j and from the work done by force component k in the j direction. The former is given by (for a unit volume)

$$T^{\bar{0}\bar{0}}v^j \tag{11}$$

since $T^{\bar{0}\bar{0}}$ is the rest mass-energy density in the object's rest frame, and the flux is the mass-energy density per unit j-direction distance per unit time, which is v^j.

The work done by force component k in moving a distance Δx^j in the j direction is force times distance, or $T^{\bar{j}\bar{k}}\Delta x^j$, and the rate at which this work is done is that divided by the time Δt, or

$$T^{\bar{j}\bar{k}}\frac{\Delta x^j}{\Delta t}=T^{\bar{j}\bar{k}}v^j \tag{12}$$

Combining these reproduces 9 and 10.

(c) As the momentum density in direction x^j is given by T^{0j}, the force per unit volume required to give an acceleration of $a^j = \frac{dv^j}{dt}$ in the x^j direction is

$$F^j = \frac{dT^{0j}}{dt} = \sum_k m^{jk} \frac{dv^k}{dt} \tag{13}$$

For a perfect fluid at rest, we have

$$T^{\bar{\mu}\bar{\nu}} = \begin{bmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{bmatrix} \tag{14}$$

where ρ is the rest mass-energy density and p is the pressure. Thus from 10 we have for the spatial components of m^{jk}:

$$m^{jk} = \begin{bmatrix} \rho+p & 0 & 0 \\ 0 & \rho+p & 0 \\ 0 & 0 & \rho+p \end{bmatrix} \tag{15}$$

$$= (\rho+p)\, I \tag{16}$$

where I is the 3×3 identity matrix. Thus the inertial mass density m^{jk} is composed of the rest mass-energy (represented by ρ) and the energy from the random motion of the particles in the fluid, represented by the pressure p.

(d) For an isolated, stressed body that is in equilibrium so that there is no net rate of change of any of the stress-energy components, that is, $T^{\alpha\beta}{}_{,0} = 0$, all forces within the body must balance out, since there can be no internal net motion. In this case, the integral of each of the momentum transfer components of T^{jk} over the whole body must come out to zero. If we define the total inertial mass by

$$M^{jk} = \int_{\text{stressed body}} m^{jk} dx\, dy\, dz \tag{17}$$

$$= \int_{\text{stressed body}} \left(T^{\bar{0}\bar{0}} \delta^{jk} + T^{\bar{j}\bar{k}} \right) dx\, dy\, dz \tag{18}$$

$$= \delta^{jk} \int_{\text{stressed body}} T^{\bar{0}\bar{0}} dx\, dy\, dz \tag{19}$$