HYPERBOLIC MOTION IN SPECIAL RELATIVITY

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References: Charles W. Misner, Kip S. Thorne & John Archibald Wheeler, *Gravitation*, W.H. Freeman (1973). Exercise 6.1.

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In section 6.2, MTW derive the formulas for describing a fixed acceleration (relative to an inertial frame) in special relativity. A rocket has 4-velocity u which satisfies

$$u\cdot u=u_{\mu}u^{\mu}=-1 \tag{1}$$

The acceleration is given by

$$a=\frac{du}{d\tau} \tag{2}$$

Taking the derivative of 1 we have

$$\frac{d}{d\tau}(u\cdot u)=2u\cdot\frac{du}{d\tau}=2u\cdot a=\frac{d}{d\tau}(-1)=0 \tag{3}$$

Thus

$$u\cdot a=0 \tag{4}$$

so that the 4-acceleration is orthogonal to the 4-velocity.

We can fill in a few details in MTW's derivation of the rest of this section. We consider a rocket that has a fixed acceleration A (I'll use A instead of MTW's g to avoid confusion with the metric tensor $g_{\mu\nu}$) in the x^1 direction. In the inertial frame, we have

$$\begin{aligned}\frac{dt}{d\tau}&=u^0\\ \frac{dx}{d\tau}&=u^1\\ \frac{du^0}{d\tau}&=a^0\\ \frac{du^1}{d\tau}&=a^1\end{aligned} \tag{5}$$

From 4 we have

$$u^\mu a_\mu = g_{\mu\nu} u^\mu a^\nu = -u^0 a^0 + u^1 a^1 = 0 \tag{6}$$

and from 1 we have

$$-\left(u^0\right)^2 + \left(u^1\right)^2 = -1 \tag{7}$$

The fixed acceleration A gives us the condition

$$a^\mu a_\mu = -\left(a^0\right)^2 + \left(a^1\right)^2 = A^2 \tag{8}$$

where A^2 is A-squared, not the 2 component of a vector.

From 6 we have

$$a^0 = \frac{u^1 a^1}{u^0} \tag{9}$$

Plug this into 8:

$$\left(a^1\right)^2 \left(-\frac{\left(u^1\right)^2}{\left(u^0\right)^2} + 1\right) = A^2 \tag{10}$$

$$\frac{\left(a^1\right)^2}{\left(u^0\right)^2}\left[-\left(u^1\right)^2 + \left(u^0\right)^2\right] = A^2 \tag{11}$$

$$a^1 = u^0 A \tag{12}$$

$$\frac{du^1}{d\tau} = u^0 A \tag{13}$$

To get the third line, we used 7 on the second line.

We can get a similar equation by inserting 12 into 8 to get

$$-\left(a^0\right)^2 + \left(u^0\right)^2 A^2 = A^2 \tag{14}$$

$$\frac{\left(a^0\right)^2}{\left(u^0\right)^2 - 1} = A^2 \tag{15}$$

From 7, $\left(u^0\right)^2 - 1 = \left(u^1\right)^2$ so we have

$$a^0 = u^1 A \tag{16}$$

$$\frac{du^0}{d\tau} = u^1 A \tag{17}$$

Equations 17 and 13 form a pair of coupled ODEs which can be solved. Insert 17 into 13 to get

$$\frac{1}{A}\frac{d}{d\tau}\left(\frac{du^0}{d\tau}\right) = u^0 A \tag{18}$$

$$\frac{d^2u^0}{d\tau^2} = A^2 u^0 \tag{19}$$

This has two independent solutions of the form

$$u^0(\tau) = \begin{cases} C_1 e^{-A\tau} \\ C_2 e^{A\tau} \end{cases} \tag{20}$$

where C_1 and C_2 are constants of integration. Thus the general solution is the sum (since the ODE is linear):

$$u^0(\tau) = C_1 e^{-A\tau} + C_2 e^{A\tau} \tag{21}$$

From 17 we can now get $u^1(\tau)$:

$$u^1(\tau) = -C_1 e^{-A\tau} + C_2 e^{A\tau} \tag{22}$$

If we impose the initial condition that at $\tau = 0$ the rocket is at rest, then $u^0 = 1$ and $u^1 = 0$ which allows us to find C_1 and C_2, and we get

$$u^0(\tau) = \frac{e^{-A\tau} + e^{A\tau}}{2} \tag{23}$$

$$= \cosh(A\tau) \tag{24}$$

$$u^1(\tau) = \frac{-e^{-A\tau} + e^{A\tau}}{2} \tag{25}$$

$$= \sinh(A\tau) \tag{26}$$

We can integrate these to get expressions for $t(\tau)$ and $x(\tau)$, and we get

$$t(\tau) = \frac{1}{A}\sinh(A\tau) + t_0 \tag{27}$$

$$x(\tau) = \frac{1}{A}\cosh(A\tau) + x_0 \tag{28}$$

where t_0 and x_0 are constants of integration which MTW take to be both zero, so we have

$$t(\tau) = \frac{1}{A}\sinh(A\tau) \tag{29}$$

$$x(\tau) = \frac{1}{A}\cosh(A\tau) \tag{30}$$

Note that these equations describe an initial position of $x(0) = \frac{1}{A}$, not zero, at $\tau = 0$.

Ex 6.1. We now consider a rocket travelling from Earth to the centre of the Milky Way galaxy, a distance of about 30,000 light years. We take the acceleration of the rocket to be one Earth gravity (now I'll have to use g for the acceleration) for the first half of the journey, and then a constant deceleration of g for the second half. The proper time as seen by a passenger on the rocket can be found from 30. We have

$$\tau = \frac{1}{g}\cosh^{-1}(gx) \tag{31}$$

To do the calculation, we need to get the units right. As shown in MTW, we can convert g to light years. The units of an acceleration are $\text{cm}\cdot\text{sec}^{-2}$ which can be converted into length alone by dividing by c^2, since the units work out to $\left(\text{cm}\cdot\text{sec}^{-2}\right) \div \left(\text{cm}^2\cdot\text{sec}^{-2}\right) = \text{cm}^{-1}$. Thus we can write g as

$$g = \frac{980\text{ cm}\cdot\text{sec}^{-2}}{\left(3\times10^{10}\text{cm}\cdot\text{sec}^{-1}\right)^2} \tag{32}$$

$$= 1.09\times10^{-18}\text{ cm}^{-1} \tag{33}$$

One light year is

$$1\text{ ly} = 9.461\times10^{17}\text{ cm} \tag{34}$$

so

$$\frac{1}{g} = 9.183\times10^{17}\text{ cm} \tag{35}$$

Thus for an approximate calculation, we can take

$$g \approx 1\text{ ly}^{-1} \tag{36}$$

From 31 we have for the first half of the journey

$$\tau = \frac{1}{1}\cosh^{-1}1.5\times10^4\times1 \tag{37}$$

$$= 10.3\text{ ly} \tag{38}$$

$$= 10.3\text{ years} \tag{39}$$

Since the second half of the journey is symmetric with the first half with constant deceleration at the same rate, the total proper time experienced by the rocket is about 20.6 years.

The travel time as measured by someone in the Earth's inertial frame can be found from 29. At the halfway point, $\tau=10.3$ years so, with $g=1$ ly^{-1} we have

$$t\left(10.3\text{ years}\right)=\sinh 10.3=1.487\times10^{4}\text{ years} \tag{40}$$

This may seem odd, since it looks like we're travelling faster than light (which takes 1.5×10^{4} years to travel 15,000 light years). However, recall from 30 that we actually have a 1 light year head start on a photon starting at $x=0$, and as MTW point out, given such a head start we can actually outrun a photon.

PINGBACKS

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