PAYLOAD OF A RELATIVISTIC ROCKET

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References: Charles W. Misner, Kip S. Thorne & John Archibald Wheeler, *Gravitation*, W.H. Freeman (1973). Exercise 6.2.

Post date: 7 Sep 2020.

In the previous post, we considered a rocket travelling to the centre of the galaxy (a distance of 30,000 light years) at a constant acceleration of 1 Earth gravity for half the journey, then a constant deceleration at the same rate for the second half. Here we'll look at how much of a payload (that is, passengers and supplies) could be carried on such a rocket. The problem says we should assume that the mass of the rocket is converted to energy in the usual way (with $E = mc^2$) and that all this energy is used to provide thrust for the rocket. In the following, I'll use light years (ly) as the distance unit and years as the time unit, so $c = 1$ ly $\cdot$ year^{-1} and $g \approx 1$ ly^{-1} (see previous post).

Let's consider the first half of the journey, and let M be the mass that is used as fuel for the first half of the journey and m be the mass of the payload, so that the initial mass (energy) is

$$E_0 = M + m \tag{1}$$

At the halfway point, M has been converted to energy E_f and, in the Earth's reference frame, the mass of the payload is now γm. By conservation of energy, this must be equal to the initial energy so we have

$$E_f + \gamma m = M + m \tag{2}$$

Now consider momentum conservation. The initial momentum is zero. During the flight, the only force acting on the rocket is due to the reaction force from the light being expelled behind the rocket, so the total momentum of rocket + expelled light must remain at zero. The momentum of a photon is, in geometric units, equal to its energy, so the momentum of the expelled photons must be E_f. The momentum of the rocket at the halfway point is $\gamma m v$, where v is its velocity as measured in the Earth frame. Thus conservation of momentum gives us

$$\gamma m v = E_f \tag{3}$$

Combining this with 2 we have

$$\frac{M}{m}=\gamma(1+v)-1 \tag{4}$$

To get some numbers, we use the equations for the 4-velocity components:

$$u^0(\tau)=\frac{e^{-g\tau}+e^{g\tau}}{2} \tag{5}$$
$$=\cosh(g\tau) \tag{6}$$
$$u^1(\tau)=\frac{-e^{-g\tau}+e^{g\tau}}{2} \tag{7}$$
$$=\sinh(g\tau) \tag{8}$$

where τ is the proper time measured on the rocket. Since

$$\begin{aligned} u^0(\tau)&=\frac{dt}{d\tau}=\gamma \\ u^1(\tau)&=\gamma v \end{aligned} \tag{9}$$

we have

$$v=\frac{u^1(\tau)}{u^0(\tau)}=\tanh(g\tau) \tag{10}$$

and

$$\frac{M}{m}=\cosh(g\tau)(1+\tanh(g\tau))-1 \tag{11}$$

At the halfway point, $\tau=10.3$ years = 10.3 ly in geometric units, and we're taking $g=1$ ly^{-1}, so we have

$$\frac{M}{m}=2.97\times10^4 \tag{12}$$

Thus for each kilogram of payload, we need about 30 metric tons of extra mass that is converted to energy to get halfway.

We can't just double this to get the payload for the entire trip, since we need to carry the extra mass that is used to decelerate in the second half of the journey all the way from the start. However, the situation in the second half is equivalent (from the point of view of how much fuel is required) to continuing to accelerate at the same rate during the second half, but still do this for the same proper time as was taken to go the first half of the journey. Thus we can just double the proper time in 11, and we find, for the entire trip with $\tau=20.6$ ly

$$\frac{M}{m} = 8.84 \times 10^8 \tag{13}$$

Note that the situation where we actually do accelerate at the same rate all the way to the centre of the galaxy (as opposed to accelerating for the first half and then decelerating for the second half) is quite different, since in that case, the proper time required to reach the centre of the galaxy is considerably less. In this case, we'd reach the galactic centre travelling at near light speed, and the proper time required for the journey is

$$\tau = \frac{1}{g}\cosh^{-1}(gx) \tag{14}$$

with $x = 3 \times 10^4$. This gives

$$\tau = 11 \text{ ly} \tag{15}$$

so the second half of the journey takes less than a year in proper time, even though the first half takes 10.3 years. The fuel ratio for this type of trip is

$$\frac{M}{m} = 6 \times 10^4 \tag{16}$$

so in this case, the amount of fuel required is roughly double that for the first half of the journey 12. The reason this is so much less than 13 is that so much less proper time is needed for the second half of the journey than in the decelerating case. As the rocket decelerates, more proper time is needed to travel the same distance (as seen in the Earth frame).