ELECTROMAGNETIC FIELD TENSOR AS EXTERIOR PRODUCTS

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References: Charles W. Misner, Kip S. Thorne & John Archibald Wheeler, *Gravitation*, W.H. Freeman (1973). Figure 4.1.

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In section 4.2, MTW define the electromagnetic field tensor in two ways. One way uses the tensor product of two one-forms and has the form

$$F = F_{\alpha\beta}\boldsymbol{d}x^{\alpha}\otimes\boldsymbol{d}x^{\beta} \tag{1}$$

The term $F_{\alpha\beta}$ is a component of the Faraday tensor F. However, the indices α and β on $\boldsymbol{d}x^{\alpha}\otimes\boldsymbol{d}x^{\beta}$ indicate a particular one-form and *not* a component of that one form. Remember that the tensor product $\boldsymbol{d}x^{\alpha}\otimes\boldsymbol{d}x^{\beta}$ is a machine that takes two vectors as inputs and outputs a number, according to

$$\boldsymbol{d}x^{\alpha}\otimes\boldsymbol{d}x^{\beta}(u,v) = \langle\boldsymbol{d}x^{\alpha},u\rangle\left\langle\boldsymbol{d}x^{\beta},v\right\rangle \tag{2}$$

In a typical coordinate system where the spatial coordinates form a right-handed orthogonal system, we have

$$\begin{aligned}\boldsymbol{d}x^{0} &= (1,0,0,0)\\ \boldsymbol{d}x^{1} &= (0,1,0,0)\\ \boldsymbol{d}x^{2} &= (0,0,1,0)\\ \boldsymbol{d}x^{3} &= (0,0,0,1)\end{aligned} \tag{3}$$

so

$$\langle\boldsymbol{d}x^{\alpha},u\rangle\left\langle\boldsymbol{d}x^{\beta},v\right\rangle = u^{\alpha}v^{\beta} \tag{4}$$

where now the indices α and β represent which one-form on the LHS, and on the RHS they *do* indicate components of the vectors u and v.

Another way of defining F uses an exterior product, which is defined by

$$\boldsymbol{d}x^{\alpha}\wedge\boldsymbol{d}x^{\beta} \equiv \boldsymbol{d}x^{\alpha}\otimes\boldsymbol{d}x^{\beta} - \boldsymbol{d}x^{\beta}\otimes\boldsymbol{d}x^{\alpha} \tag{5}$$

In terms of exterior products, any rank-2 antisymmetric tensor F can be defined as

$$F = \frac{1}{2} F_{\alpha\beta} \boldsymbol{d}x^{\alpha} \wedge \boldsymbol{d}x^{\beta} \tag{6}$$

In the early sections of Chapter 4, and in particular in Fig. 4.1 and Box 4.2, MTW describe how the exterior product can be visualized as a honeycomb type structure with a given circulation. I won't repeat the discussion here, as a bit of study of their book should give the reader a good idea of how this works. Basically, an exterior product defines a structure which looks like tubes or lines extending through space. This structure constitues a 2-form, which is a machine for constructing a number out of an oriented surface. This surface is pictured as intersecting the 2-form's tubes over some particular area, and the number constructed by this is the integral of the tensor F over the surface.

At the end of Fig. 4.1 they pose an exercise for the reader. We are to derive an expression for the number constructed by doing this integration. The surface that intersects the 2-form is defined by taking two vectors u and v and forming a parallelogram out of them. From basic linear algebra, we have the formula for the area of such a parallelogram in 3-d space:

$$A = |\boldsymbol{u} \times \boldsymbol{v}| \tag{7}$$

That is, we calculate the usual vector cross product of the two 3-d vectors and take the magnitude.

In general, it appears that the area of a parallelogram is given by the exterior product, as in

$$A = u \wedge v \tag{8}$$

However, given the definition 5, it would appear that in MTW's book, this area is given instead by half this, or

$$A = \frac{1}{2} u \wedge v \tag{9}$$

I'm not entirely clear on how MTW's definition compares to other mathematical definitions, since most of the articles are too mathematically technical for me to follow. However, it seems that 9 is necessary for the following derivation to work out. It also seems reasonable, since in 3-d, the area could also be written as

$$A = \frac{1}{2} (\boldsymbol{u} \times \boldsymbol{v} - \boldsymbol{v} \times \boldsymbol{u}) \tag{10}$$

For more details on this definition of area and its relation to the 3-d cross product, see the Wikipedia article on exterior algebra.

Anyway, I'll take 9 as the working definition in what follows. Starting with 6, we want to find the integral of F over an infinitesimal area bounded by the parallelogram whose borders are given by u and v. Since we're dealing with infinitesimals, we can approximate the integral by just multiplying F by the area of the parallelogram, so we have

$$\int_{u\wedge v} F \approx F\left(\frac{1}{2}u\wedge v\right) \tag{11}$$

$$= \frac{1}{2}F_{\alpha\beta}\left(\boldsymbol{d}x^{\alpha}\wedge\boldsymbol{d}x^{\beta}\right)\left(\frac{1}{2}u\wedge v\right) \tag{12}$$

$$= \frac{1}{4}F_{\alpha\beta}\left(\boldsymbol{d}x^{\alpha}\otimes\boldsymbol{d}x^{\beta}-\boldsymbol{d}x^{\beta}\otimes\boldsymbol{d}x^{\alpha}\right)(u\otimes v-v\otimes u) \tag{13}$$

The components of a tensor product $T = u\otimes v$ are given by

$$T^{\mu\nu} = u^{\mu}v^{\nu} \tag{14}$$

so in terms of components we can write 13 as

$$\int_{u\wedge v} F \approx \frac{1}{4}F_{\alpha\beta}\left(\boldsymbol{d}x^{\alpha}_{\mu}\boldsymbol{d}x^{\beta}_{\nu}-\boldsymbol{d}x^{\beta}_{\mu}\boldsymbol{d}x^{\alpha}_{\nu}\right)\left(u^{\mu}v^{\nu}-v^{\mu}u^{\mu}\right) \tag{15}$$

$$= \frac{1}{4}F_{\alpha\beta}\left(u^{\alpha}v^{\beta}-v^{\alpha}u^{\beta}-u^{\beta}v^{\alpha}+v^{\beta}u^{\alpha}\right) \tag{16}$$

$$= \frac{1}{2}F_{\alpha\beta}\left(u^{\alpha}v^{\beta}-u^{\beta}v^{\alpha}\right) \tag{17}$$

$$= \frac{1}{2}\left(F_{\alpha\beta}u^{\alpha}v^{\beta}+F_{\beta\alpha}u^{\beta}v^{\alpha}\right) \tag{18}$$

$$= F_{\alpha\beta}u^{\alpha}v^{\beta} \tag{19}$$

In the first line, the term $\boldsymbol{d}x^{\alpha}_{\mu}$ indicates the μth component of the one-form $\boldsymbol{d}x^{\alpha}$ where (remember) the α indicates which one-form we're using, and is not a component index. The tensor product $\boldsymbol{d}x^{\alpha}\otimes\boldsymbol{d}x^{\beta}$ results in a 4×4 matrix whose $\mu\nu$ element is $\boldsymbol{d}x^{\alpha}_{\mu}\boldsymbol{d}x^{\beta}_{\nu}$. Similarly, $u\otimes v$ is a 4×4 matrix with components $u^{\mu}v^{\nu}$. By looking at 3, we see that the summed product $\boldsymbol{d}x^{\alpha}_{\mu}u^{\mu}$ gives

$$\boldsymbol{d}x^{\alpha}_{\mu}u^{\mu} = u^{\alpha} \tag{20}$$

The second line follows from using 3 and doing the sums over μ and ν. The fourth line follows from the antisymmetry of $F_{\alpha\beta} = -F_{\beta\alpha}$, and the last line follows from swapping $\alpha\leftrightarrow\beta$ in the second term of the fourth line.

This result is equivalent to the earlier form of F as a rank-2 tensor with two slots, as we can write

$$F(u,v) = F(u^{\mu}\boldsymbol{e}_{\mu}, v^{\nu}\boldsymbol{e}_{\nu}) \tag{21}$$
$$= u^{\mu}v^{\nu}F(\boldsymbol{e}_{\mu}, \boldsymbol{e}_{\nu}) \tag{22}$$
$$= F_{\mu\nu}u^{\mu}v^{\nu} \tag{23}$$

This follows because the definition of the components of a tensor are defined as the results of its action on the basis vectors $\boldsymbol{e}_{\mu}$.