FOUR-VELOCITY: AN EXAMPLE

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We’ve already looked at the four-velocity in special relativity, but it’s worth a second look from a different angle. We can instead define the four-velocity in terms of two events separated by an infinitesimal spacetime interval $ds$. The four-velocity is defined as the derivative of $ds$ with respect to the proper time $\tau$, so that

$$u^i \equiv \frac{d s^i}{d \tau} \quad (1)$$

Since the components $d s^i$ transform using the Lorentz transformation, then so do the components $u^i$ of the four-velocity.

Since the spacetime interval is invariant (it has the same value in all inertial frames) the relation

$$d s^2 = -d t^2 + d x^2 + d y^2 + d z^2 \quad (2)$$

holds in all inertial frames. In particular, it holds in the observer’s rest frame, in which $d t = d \tau$, so we have

$$d s^2 = -d \tau^2 \quad (3)$$

In this rest frame, then, we get

$$u^i = \left( \frac{d \tau}{d \tau}, 0, 0, 0 \right) \quad (4)$$

$$= (1, 0, 0, 0) \quad (5)$$

The square of $u$ is then

$$u \cdot u = \eta_{ij} u^i u^j \quad (6)$$

$$= -1 \quad (7)$$

where $\eta_{ij}$ is the metric used in special relativity:
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\[ \eta_{ij} = \begin{cases} 
-1 & i = j = t \\
1 & i = j = x, y, z \\
0 & \text{otherwise}
\end{cases} \quad (8) \]

Since this is true in the rest frame and the square of a four-vector is an invariant, it is true in all frames.

As an example, suppose we have an object that moves along a worldline given by (in some inertial frame)

\[ x(\tau) = \frac{1}{g} \left[ \cosh (g\tau) - 1 \right] \quad (9) \]

\((y = z = 0 \text{ and } g \text{ is a constant})\). The \(x\) component of the four-velocity is then

\[ u^x = \frac{dx}{d\tau} \quad (10) \]

\[ = \sinh (g\tau) \quad (11) \]

Using \(u \cdot u = -1\) we can find the \(t\) component:

\[ (u^x)^2 - (u^t)^2 = -1 \quad (12) \]

\[ \sinh^2 (g\tau) + 1 = (u^t)^2 \quad (13) \]

\[ u^t = \cosh (g\tau) \quad (14) \]

using the identity \(\cosh^2 x - \sinh^2 x = 1\). From this we can get the time in the inertial frame:

\[ u^t = \frac{dt}{d\tau} \quad (15) \]

\[ t(\tau) = \frac{1}{g} \sinh (g\tau) \quad (16) \]

The velocity of the object as seen in the inertial frame is

\[ v_x = \frac{dx}{dt} \quad (17) \]

\[ = \frac{u^x}{u^t} \quad (18) \]

\[ = \tanh (g\tau) \quad (19) \]
Since tanh is bounded by $\pm 1$, the velocity never exceeds 1, so never exceeds the speed of light.

We can invert the relation between proper time $\tau$ and inertial time $t$ to get

$$g\tau = \sinh^{-1}(gt)$$  \hspace{1cm} (20)

Using the relations (derived from $\cosh^2 x - \sinh^2 x = 1$)

$$\sinh (\sinh^{-1} x) = x$$  \hspace{1cm} (21)
$$\cosh (\sinh^{-1} x) = \sqrt{1+x^2}$$  \hspace{1cm} (22)
$$\tanh (\sinh^{-1} x) = \frac{x}{\sqrt{1+x^2}}$$  \hspace{1cm} (23)

we get

$$u^x(t) = gt$$  \hspace{1cm} (24)
$$u^t(t) = \sqrt{1+(gt)^2}$$  \hspace{1cm} (25)
$$v(t) = \frac{gt}{\sqrt{1+(gt)^2}}$$  \hspace{1cm} (26)

Again, note that $u \cdot u = -1$ and also that as $t \to \infty$, $v \to 1$ so again the velocity remains less than $c$.

PINGBACKS

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