METRIC TENSOR UNDER LORENTZ TRANSFORMATION

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The invariant interval in special relativity can be written as

$$ds^2 = \eta_{ij} dx^i dx^j$$  \hspace{1cm} (1)

where $\eta_{ij}$ is the metric tensor in flat space, with components $\eta_{00} = -1$, $\eta_{ii} = +1$ for $i = 1, 2, 3$ and zero otherwise. Thus this relation is the same as

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2$$  \hspace{1cm} (2)

Under a Lorentz transformation, we get

$$ds^2 = \eta_{ij} dx'^i dx'^j$$  \hspace{1cm} (3)

$$= \eta_{ij} \Lambda^i_a \Lambda^j_b dx^a dx^b$$  \hspace{1cm} (4)

Since the interval is invariant, we get

$$\eta_{ij} \Lambda^i_a \Lambda^j_b dx^a dx^b = \eta_{ab} dx^a dx^b$$  \hspace{1cm} (5)

$$\left( \eta_{ij} \Lambda^i_a \Lambda^j_b - \eta_{ab} \right) dx^a dx^b = 0$$  \hspace{1cm} (6)

Since the last equation must be true for an infinitesimal interval, the quantity in parentheses must be zero, so

$$\eta_{ab} = \eta_{ij} \Lambda^i_a \Lambda^j_b$$  \hspace{1cm} (7)

That is, if we apply a Lorentz transformation (the same transformation!) to each index in the metric tensor, we get the same tensor back again.

We can multiply this equation by an inverse transformation to get

$$\left( \Lambda^{-1} \right)_k^a \eta_{ab} = \eta_{ij} \Lambda^i_a \Lambda^j_b \left( \Lambda^{-1} \right)_k^a$$  \hspace{1cm} (8)

Multiplying a transformation by its inverse gives the identity matrix:

$$\Lambda^i_a \left( \Lambda^{-1} \right)_k^a = \delta^i_k$$  \hspace{1cm} (9)
So we get

\[(\Lambda^{-1})^a_k \eta_{ab} = \eta_{ij} \delta^i_k \Lambda^j_b \quad (10)\]
\[= \eta_{kj} \Lambda^j_b \quad (11)\]

Repeating the process, we get

\[\left(\Lambda^{-1}\right)^b_l \left(\Lambda^{-1}\right)^a_k \eta_{ab} = \eta_{kj} \Lambda^j_b \left(\Lambda^{-1}\right)^b_l \quad (12)\]
\[= \eta_{kj} \delta^j_l \quad (13)\]
\[= \eta_{kl} \quad (14)\]

Thus, not surprisingly, if we multiply the metric tensor by two inverse Lorentz transformations, we get the same tensor back.

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