ELECTROMAGNETIC FIELD TENSOR: CONSERVATION OF MASS

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As we’ll study in more detail a bit later, the electric and magnetic fields can be combined into a single tensor known as the electromagnetic field tensor \( F^{ij} \):

\[
F^{ij} = \begin{bmatrix}
0 & E_x & E_y & E_z \\
-E_x & 0 & B_z & -B_y \\
-E_y & -B_z & 0 & B_x \\
-E_z & B_y & -B_x & 0
\end{bmatrix}
\] (1)

We can see from its definition that this tensor is anti-symmetric, that is, that \( F^{ij} = -F^{ji} \). For any anti-symmetric tensor we can show that

\[
F^{ij} \eta_{ia} \eta_{jb} u^a u^b = 0
\] (2)

In this equation, \( \eta_{ij} \) is the metric tensor in flat space and \( u^a \) is the four-velocity, but in fact the formula is valid for any tensors \( \eta \) and \( u \), provided that \( F \) is anti-symmetric. The proof involves a bit of index-switching.

\[
F^{ij} \eta_{ia} \eta_{jb} u^a u^b = -F^{ji} \eta_{ia} \eta_{jb} u^a u^b
\] (3)

\[
= -F^{ij} \eta_{ja} \eta_{ib} u^a u^b
\] (4)

\[
= -F^{ij} \eta_{jb} \eta_{ia} u^b u^a
\] (5)

In the second line, we swapped the dummy indexes \( i \) and \( j \), and in the third line we swapped \( a \) and \( b \). The result shows that the original quantity is equal to its negative, which means it must be zero.

In terms of \( F^{ij} \), the electric and magnetic (Lorentz) force laws for a charge \( q \) can be combined into a single equation:

\[
\frac{dp^i}{d\tau} = qF^{ij} \eta_{ia} u^a
\] (6)

where \( u^a = \gamma [1, v_x, v_y, v_z] \) is the four-velocity.
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For example, if \( i = 1 \) we get

\[
\frac{dp^1}{d\tau} = q\gamma (E_x + v_y B_z - v_z B_y)
\]  

(7)

In the non-relativistic limit, \( \gamma \to 1 \) and this is the \( x \) component of the force law \( \frac{dp}{dt} = qE + qv \times B \). We’ll explore some of the other properties of this tensor later.

Since the square of the four-momentum of a particle is the negative of its mass squared \( (p \cdot p = \gamma^2 m^2 (-1 + v^2) = -m^2) \), this should be conserved for a charged particle moving in an electromagnetic field. (Its total momentum is, of course, not conserved since the fields exert a force on the particle.)

We have

\[
\frac{d(p \cdot p)}{d\tau} = \frac{d}{d\tau} \left( \eta_{ij} p^i p^j \right) 
= \eta_{ij} \left[ \frac{dp^i}{d\tau} p^j + p^i \frac{dp^j}{d\tau} \right] 
= 2\eta_{ij} \frac{dp^i}{d\tau} p^j 
= 2q\eta_{ij} F^{ik} \eta_{ka} u^a p^j 
= 2qm F^{ik} \eta_{ij} \eta_{ka} u^a u^j 
= 0
\]

(8)  

(9)  

(10)  

(11)  

(12)  

(13)

In the third line, we used the fact that \( \eta_{ij} = \eta_{ji} \) and swapped \( i \) and \( j \) in the second term. The fourth line uses \( 6 \) and the last line uses \( 2 \).

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