VECTORS AND THE METRIC TENSOR

We can define a general vector $\mathbf{A}$ in terms of the basis vectors $\mathbf{e}_i$ in a given coordinate system:

$$\mathbf{A} \equiv A^i \mathbf{e}_i$$

This is analogous to the definition of the infinitesimal displacement that we met earlier: $d\mathbf{s} = dx^i \mathbf{e}_i$. This has a couple of consequences. First, since the basis vectors are not necessarily either unit vectors or orthogonal, this definition may be different from the usual definition of a vector that you’re used to from linear algebra courses.

Second, we require the transformation of a vector’s components between coordinate systems to be the same as the components of $d\mathbf{s}$, which means that

$$A'^i = \frac{\partial x'^i}{\partial x^j} A^j$$

Finally, the square of a vector follows the same pattern as the square of the increment $ds^2$:

$$A^2 = \mathbf{A} \cdot \mathbf{A} = g_{ij} A^i A^j$$

As an example, consider the case of uniform circular motion. From elementary physics, we know that, in polar coordinates, the radial component of the velocity $v^r = 0$ (since the object is always at the same distance from the origin) and the tangential component is $v^\theta = v$. Using the metric for polar coordinates, this means that

$$v^2 = g_{ij} v^i v^j = 0 \times 0 + r^2 \times v^\theta \times v^\theta = \left( v^\theta \right)^2 r^2$$

$$v^\theta = \frac{v}{r}$$
To transform this vector to rectangular coordinates, we have

\[ v'_{i} = \frac{\partial x'^{i}}{\partial x^{j}} v^{j} \]  

(8)

where the primed system is rectangular and the unprimed is polar. So

\[ v^{x} = \frac{\partial x}{\partial r} v^{r} + \frac{\partial x}{\partial \theta} v^{\theta} \]  

(9)

\[ = -r \sin \theta \frac{v}{r} \]  

(10)

\[ = -v \sin \theta \]  

(11)

\[ v^{y} = \frac{\partial y}{\partial r} v^{r} + \frac{\partial y}{\partial \theta} v^{\theta} \]  

(12)

\[ = r \cos \theta \frac{v}{r} \]  

(13)

\[ = v \cos \theta \]  

(14)

The square is invariant, since using the rectangular metric

\[ v^{2} = g_{ij} v^{i} v^{j} \]  

(15)

\[ = (-v \sin \theta)^{2} + (v \cos \theta)^{2} \]  

(16)

\[ = v^{2} \]  

(17)

Now let’s look at the inverse problem. This time we have an object moving at a constant speed \( v \) in the \(+y\) direction, so that \( v^{x} = 0, v^{y} = v \). To convert this to polar coordinates, we need the derivatives

\[ \frac{\partial r}{\partial x} = \frac{x}{\sqrt{x^{2} + y^{2}}} \]  

(18)

\[ \frac{\partial r}{\partial y} = \frac{y}{\sqrt{x^{2} + y^{2}}} \]  

(19)

\[ \frac{\partial \theta}{\partial x} = -\frac{y}{x^{2} + y^{2}} \]  

(20)

\[ \frac{\partial \theta}{\partial y} = \frac{x}{x^{2} + y^{2}} \]  

(21)

Then we get

\[ \frac{\partial \theta}{\partial x} = -\frac{y}{\sqrt{x^{2} + y^{2}}} \]  

\[ \frac{\partial \theta}{\partial y} = \frac{x}{\sqrt{x^{2} + y^{2}}} \]
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\[ v^r = \frac{y}{\sqrt{x^2 + y^2}} v \]  \( (22) \)
\[ = v \sin \theta \]  \( (23) \)
\[ v^\theta = \frac{x}{\sqrt{x^2 + y^2}} v \]  \( (24) \)
\[ = \frac{\cos \theta}{r} v \]  \( (25) \)

If the object starts at \((x, y) = (b, 0)\) at \(t = 0\), then \(y(t) = vt\) and \(x(t) = b\).

In polar coordinates we get

\[ r(t) = \sqrt{b^2 + (vt)^2} \]  \( (26) \)
\[ \theta(t) = \arctan \frac{vt}{b} \]  \( (27) \)
\[ v^r = \frac{vt}{\sqrt{b^2 + (vt)^2}} v \]  \( (28) \)
\[ v^\theta = \frac{b}{b^2 + (vt)^2} v \]  \( (29) \)

At \(t = 0\), the motion is entirely in the \(\theta\) direction, since the object is moving tangent to the circle \(r = b\) at that time. As time increases, the motion gradually transfers over to the radial direction, with \(\lim_{t \to \infty} v^r = v\).

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