METRIC TENSOR: SEMI-LOG COORDINATES

As an example of a different coordinate system, we can define the semi-log system by introducing coordinates $p$ and $q$ defined as:

$$ p = x $$
$$ q = e^{by} $$

where $b$ is a constant. If $x$ and $y$ have units of length, then $b$ must have units of length$^{-1}$. Note that this means that $p$ and $q$ have different units, with $p$ having the units of length and $q$ being dimensionless.

The curves of constant $p$ are just the same as the curves of constant $x$, that is, vertical lines. The curves of constant $q$ are defined by $e^{by} = k$ or $y = \ln\left(\frac{k}{b}\right)$. These are horizontal lines, although the spacing for equal steps in $k$ will translate into the variable spacing seen on semi-log plots.

If an object has an acceleration $a = a^i$ in the rectangular system, then we can find its acceleration in the semi-log system in the usual way.

$$ a^p = \frac{\partial p}{\partial x} a^x + \frac{\partial p}{\partial y} a^y $$
$$ = a^x $$

$$ a^q = \frac{\partial q}{\partial x} a^x + \frac{\partial q}{\partial y} a^y $$
$$ = be^{by} a^y $$

The units of $a^p$ are still those of acceleration, but the units of $a^q$ are length$^{-1}\cdot$acceleration.

We can work out the metric of the semi-log system from the rectangular metric by direct calculation:

$$ g'_{ij} = g_{kl} \frac{\partial x^k}{\partial x'^i} \frac{\partial x^l}{\partial x'^j} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \frac{1}{(by)^2} \end{bmatrix} $$

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The length squared of $a$ is an invariant, since

$$a^2 = g'_{ij} a'^i a'^j$$  \hspace{1cm} (8)

$$= \left( a^x \right)^2 + \left( \frac{1}{bq} \right)^2 \left( be^y a^y \right)^2$$  \hspace{1cm} (9)

$$= \left( a^x \right)^2 + (a^y)^2$$  \hspace{1cm} (10)

The basis vectors in the semi-log system have lengths obtainable from the metric:

$$\mathbf{e}_p \cdot \mathbf{e}_p = g_{pp} = 1$$  \hspace{1cm} (11)

$$|\mathbf{e}_p| = 1$$  \hspace{1cm} (12)

$$\mathbf{e}_q \cdot \mathbf{e}_q = g_{qq} = \frac{1}{(bq)^2}$$  \hspace{1cm} (13)

$$|\mathbf{e}_q| = \frac{1}{bq}$$  \hspace{1cm} (14)

Incidentally, the question part (e) as written in Moore’s book doesn’t make sense; he asks for the length of the basis vector $\partial x$. If we want to use partial derivatives as basis vectors, we need to define a curve along which to take the derivative. Since Moore doesn’t even mention the use of partial derivatives as a basis in the text, I can only assume this is a typo.

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