The inverse metric tensor $g^{ij}$ is defined so that

$$g^{ij} g_{jk} = \delta^i_k \quad (1)$$

If the metric tensor is viewed as a matrix, then this is equivalent to saying $[g^{ij}] = [g_{ij}]^{-1}$. The transformation property of $g^{ij}$ can be worked out by direct calculation, using the transformation of $g_{ij}$ and the fact that $\delta^i_k$ is invariant.

$$g'^{ij} g'_{jk} = \delta^i_k \quad (2)$$

$$= g'^{ij} \frac{\partial x^l}{\partial x'^i} \frac{\partial x^m}{\partial x'^j} g_{lm} \quad (3)$$

We can try the transformation

$$g'^{ij} = \frac{\partial x'^{i}}{\partial x^a} \frac{\partial x'^{j}}{\partial x^b} g^{ab} \quad (4)$$

Substituting, we get

$$g'^{ij} g'_{jk} = \frac{\partial x'^{i}}{\partial x^a} \frac{\partial x'^{j}}{\partial x^b} g^{ab} \frac{\partial x^l}{\partial x'^i} \frac{\partial x^m}{\partial x'^j} g_{lm} \quad (5)$$

$$= \frac{\partial x'^{i}}{\partial x^a} g^{ab} \delta^l_b \frac{\partial x^m}{\partial x'^j} g_{lm} \quad (6)$$

$$= \frac{\partial x'^{i}}{\partial x^a} g^a l \frac{\partial x^m}{\partial x'^j} g_{lm} \quad (7)$$

$$= \frac{\partial x'^{i}}{\partial x^a} \frac{\partial x^m}{\partial x'^k} \delta^a_m \quad (8)$$

$$= \delta^i_k \quad (10)$$
On line 2 we used \( \frac{\partial x^j}{\partial x^b} \frac{\partial x^l}{\partial x'^j} = \delta^l_b \) and on line 4 we used \( g^{al} g_{lm} = \delta^a_m \). Thus \( g^{ij} \) is a rank-2 contravariant tensor, and is the inverse of \( g_{ij} \) which is a rank-2 covariant tensor. Since the matrix inverse is unique (basic fact from matrix algebra), we can use the standard techniques of matrix algebra to calculate the inverse.

In rectangular coordinates, \( g^{ij} = g_{ij} \) since the metric is diagonal with all diagonal elements equal to 1. In polar coordinates in 2-d,

\[
g_{ij} = \begin{bmatrix} 1 & 0 \\ 0 & r^2 \end{bmatrix}
\]

so the inverse is

\[
g^{ij} = \begin{bmatrix} 1 & 0 \\ 0 & r^{-2} \end{bmatrix}
\]

A contravariant vector \( v^i \) can be lowered (converted to a covariant vector) by multiplying by \( g_{ij} \):

\[
v_i = g_{ij} v^j
\]

The covariant vector can be converted back into a contravariant vector by raising its index:

\[
g^{ij} v_j = g^{ij} g_{jk} v^k = \delta^i_k v^k = v^i
\]

If we start with a vector \( v^i \) in rectangular coordinates, we can convert it to polar coordinates:

\[
v^r = v^x \cos \theta + v^y \sin \theta
\]

\[
v^\theta = -v^x \frac{\sin \theta}{r} + v^y \frac{\cos \theta}{r}
\]

We can lower these components by multiplying by \( g_{ij} \)

\[
v_r = v^x \cos \theta + v^y \sin \theta
\]

\[
v_\theta = r \left( -v^x \frac{\sin \theta}{r} + v^y \frac{\cos \theta}{r} \right)
\]

The square magnitude is
\[ v^i v_i = v^r v_r + v^\theta v_\theta \]  \hspace{1cm} (22)

\[ = (v^x \cos \theta + v^y \sin \theta)^2 + (-v^x \sin \theta + v^y \cos \theta)^2 \]  \hspace{1cm} (23)

\[ = (v^x)^2 + (v^y)^2 \]  \hspace{1cm} (24)

(No implied sum on the RHS in line 1.)

PINGBACKS

Pingback: Gradient as covector: example in 2-d
Pingback: Tensor trace
Pingback: Metric tensor: trace