ELECTROMAGNETIC FIELD TENSOR: INVARIANCE OF INNER PRODUCT

Link to: physicspages home page.
To leave a comment or report an error, please use the auxiliary blog.
Post date: 27 Mar 2013.

We’ve worked out earlier the scalar quantity \( F_{ij} F^{ij} = 2 (B^2 - E^2) \). We can check this for the case of a Lorentz transformation in flat space-time.

We found there that

\[
E'_x = E_x \quad (1) \\
E'_y = \gamma E_y - \gamma \beta B_z \quad (2) \\
E'_z = \gamma E_z + \gamma \beta B_y \quad (3) \\
B'_x = B_x \quad (4) \\
B'_y = -\gamma \beta E_y + \gamma B_z \quad (5) \\
B'_z = -\gamma \beta E_z - \gamma B_y \quad (6) \\
\]

Calculating the invariant in the new system we get

\[
B'^2 - E'^2 = B_x'^2 + B_y'^2 + B_z'^2 + E_x'^2 + E_y'^2 + E_z'^2 \\
= B_x^2 + (-\gamma \beta E_y + \gamma B_z)^2 + (-\gamma \beta E_z - \gamma B_y)^2 - \\
E_x'^2 - (\gamma E_y - \gamma \beta B_z)^2 - (\gamma E_z + \gamma \beta B_y)^2 \\
= B_x^2 + (B_y^2 + B_z^2) \gamma^2 (1 - \beta^2) - E_x^2 - (E_y^2 + E_z^2) \gamma^2 (1 - \beta^2) \\
= B^2 - E^2 \quad (11)
\]

since \( \gamma = 1/\sqrt{1 - \beta^2} \). All the cross terms involving the product of a component of \( E \) and one of \( B \) cancel out between lines 2/3 and 4.