SCHWARZSCHILD METRIC: GRAVITATIONAL REDSHIFT

We can use this relation to derive a formula for the gravitational redshift. The key to this is that the wavelength of light is \( \lambda = c \Delta \tau \), where \( c = 1 \) is the speed of light and \( \Delta \tau \) is the time interval as measured by an observer required for a single wavelength to be emitted or received. Why \( \Delta \tau \) instead of \( \Delta t \)? As far as I understand it, this is because \( \tau \) is the only correct measure of time for an object at rest. The Schwarzschild \( t \) coordinate, although it’s called the ‘time coordinate’, isn’t really a measure of time directly.

The redshift arises because if we emit a light beam at \( r = r_E \) in a direction radially outwards from the mass \( M \) and receive the light beam at \( r = r_R > r_E \), then the interval \( \Delta t \) required for the passage of a single wavelength must be the same at both the emitter and the receiver. Why? Because the metric doesn’t depend on \( t \).

The proper time interval, however, is not the same at the two points because of the relation above. Plugging in the values, we get

\[
\frac{\lambda_R}{\lambda_E} = \frac{\Delta \tau_R}{\Delta \tau_E} = \sqrt{\frac{1 - 2GM/r_R}{1 - 2GM/r_E}} \tag{2}
\]

This formula could be used, for example, to calculate the redshift due to a star when observed from Earth.

If both distances are large compared to \( 2GM \), we can expand the formula in a series up to first order:
\[ \frac{\lambda_R}{\lambda_E} \approx \left(1 - \frac{GM}{r_R} \right) \left(1 + \frac{GM}{r_E} \right) \]
\[ = 1 + GM \left( \frac{1}{r_E} - \frac{1}{r_R} \right) + \ldots \]
(4)

As a further approximation, if the distance \( h = r_R - r_E \ll r_E \), that is, the distance between emission and reception is small compared with the radial coordinate, then we can write

\[ \frac{\lambda_R}{\lambda_E} \approx 1 + GM \left( \frac{1}{r_E} - \frac{1}{r_R} \right) \]
\[ = 1 + GM \left( \frac{h}{r_E r_R} \right) \]
\[ \approx 1 + \frac{GM}{r^2} h \]
(5)

where \( r \) in the last line can be taken as the average of \( r_R \) and \( r_E \). In this limit, we’d expect Newton’s law of gravitation to apply, and a particle a distance \( r \) from a mass \( M \) experiences an acceleration \( g = GM/r^2 \), so we have

\[ \frac{\lambda_R}{\lambda_E} \approx 1 + gh \]
(6)

As an example, suppose we have a neutron star with mass \( M = 3 \times 10^{30} \text{ kg} \) and Schwarzschild radial coordinate at the surface of \( r_E = 1.2 \times 10^4 \text{ m} \). The redshift observed by a satellite orbiting the star at a radius \( r_R = 1.7 \times 10^4 \text{ m} \) can be calculated using the approximation formula. We need to express \( G \) in relativistic units (that is, where \( c = 1 \) so that \( GM \) has the units of length). Since the units of \( G \) are \( \text{m}^3\text{kg}^{-1}\text{s}^{-2} \), we need to eliminate the reference to seconds which we can do by dividing by \( c^2 = 9 \times 10^{16} \text{m}^2\text{s}^{-2} \). That is

\[ G = \frac{6.67 \times 10^{-11}}{9 \times 10^{16}} \]
\[ = 7.41 \times 10^{-28} \text{ m kg}^{-1} \]
(7)

For the neutron star,
\[ G M = \left(7.41 \times 10^{-28}\right) \left(3 \times 10^{30}\right) \]  
\[ = 2.223 \times 10^3 \text{ m} \] \hfill (12)

We take

\[ r = \frac{1}{2} (r_E + r_R) \] \hfill (14)
\[ = 1.45 \times 10^4 \text{ m} \] \hfill (15)

\[ g = \frac{G M}{r^2} \] \hfill (16)
\[ = \frac{2.223 \times 10^3}{(1.45 \times 10^4)^2} \] \hfill (17)
\[ = 1.06 \times 10^{-5} \text{ m}^{-1} \] \hfill (18)

Incidentally, this is a massive acceleration compared to that on the Earth’s surface. In SI units, this comes out to \( (1.06 \times 10^{-5}) (9 \times 10^{16}) = 9.54 \times 10^{11} \text{ m s}^{-2} \).

The fractional redshift is

\[ \frac{\lambda_R - \lambda_E}{\lambda_E} \approx g (r_R - r_E) \] \hfill (19)
\[ = \left(1.06 \times 10^{-5}\right) (5 \times 10^3) \] \hfill (20)
\[ = 0.0529 \] \hfill (21)

The exact value is

\[ \frac{\lambda_R - \lambda_E}{\lambda_E} = \sqrt{\frac{1 - 2GM/r_R}{1 - 2GM/r_E}} - 1 \] \hfill (22)
\[ = 0.0831 \] \hfill (23)

This is the gravitational redshift formula. For \( r_R \to \infty \), the formula reduces to

\[ \frac{\lambda_R}{\lambda_E} = \frac{1}{\sqrt{1 - 2GM/r_E}} \] \hfill (24)

This formula could be used, for example, to calculate the redshift due to a star when observed from Earth.

If both distances are large compared to \( 2GM \), we can expand the formula in a series up to first order:
\[ \frac{\lambda_R}{\lambda_E} \approx \left( 1 - \frac{GM}{r_R} \right) \left( 1 + \frac{GM}{r_E} \right) \] (25)
\[ = 1 + GM \left( \frac{1}{r_E} - \frac{1}{r_R} \right) + \ldots \] (26)

As a further approximation, if the distance \( h = r_R - r_E \ll r_E \), that is, the distance between emission and reception is small compared to the radial coordinate, then we can write

\[ \frac{\lambda_R}{\lambda_E} \approx 1 + GM \left( \frac{1}{r_E} - \frac{1}{r_R} \right) \] (27)
\[ = 1 + GM \left( \frac{h}{r_E r_R} \right) \] (28)
\[ \approx 1 + \frac{GM}{r^2} h \] (29)

where \( r \) in the last line can be taken as the average of \( r_R \) and \( r_E \). In this limit, we’d expect Newton’s law of gravitation to apply, and a particle a distance \( r \) from a mass \( M \) experiences an acceleration \( g = GM/r^2 \), so we have

\[ \frac{\lambda_R}{\lambda_E} \approx 1 + gh \] (30)

As an example, suppose we have a neutron star with mass \( M = 3 \times 10^{30} \text{ kg} \) and Schwarzschild radial coordinate at the surface of \( r_E = 1.2 \times 10^4 \text{ m} \). The redshift observed by a satellite orbiting the star at a radius \( r_R = 1.7 \times 10^4 \text{ m} \) can be calculated using the approximation formula. We need to express \( G \) in relativistic units (that is, where \( c = 1 \) so that \( GM \) has the units of length). Since the units of \( G \) are \( \text{m}^3\text{kg}^{-1}\text{s}^{-2} \), we need to eliminate the reference to seconds which we can do by dividing by \( c^2 = 9 \times 10^{16} \text{m}^2\text{s}^{-2} \). That is

\[ G = \frac{6.67 \times 10^{-11}}{9 \times 10^{16}} \]
\[ = 7.41 \times 10^{-28} \text{ m kg}^{-1} \] (32)

For the neutron star,
\[ G M = \left(7.41 \times 10^{-28}\right) \left(3 \times 10^{30}\right) \]  \hfill (33)
\[ = 2.223 \times 10^3 \text{ m} \] \hfill (34)

We take

\[ r = \frac{1}{2} \left(r_E + r_R\right) \] \hfill (35)
\[ = 1.45 \times 10^4 \text{ m} \] \hfill (36)

\[ g = \frac{GM}{r^2} \] \hfill (37)
\[ = \frac{2.223 \times 10^3}{\left(1.45 \times 10^4\right)^2} \] \hfill (38)
\[ = 1.06 \times 10^{-5} \text{ m}^{-1} \] \hfill (39)

Incidentally, this is a massive acceleration compared to that on the Earth’s surface. In SI units, this comes out to \(1.06 \times 10^{-5} \left(9 \times 10^{16}\right) = 9.54 \times 10^{11}\text{ m s}^{-2}\).

The fractional redshift is

\[ \frac{\lambda_R - \lambda_E}{\lambda_E} \approx g \left(r_R - r_E\right) \] \hfill (40)
\[ = \left(1.06 \times 10^{-5}\right) \left(5 \times 10^3\right) \] \hfill (41)
\[ = 0.0529 \] \hfill (42)

The exact value is

\[ \frac{\lambda_R - \lambda_E}{\lambda_E} = \sqrt{\frac{1 - 2GM/r_R}{1 - 2GM/r_E}} - 1 \] \hfill (43)
\[ = 0.0831 \] \hfill (44)

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