PARTICLE FALLING TOWARDS A MASS: TWO TYPES OF VELOCITY

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The radial equation of motion in the Schwarzschild metric is

\[
\frac{1}{2} \left( \frac{dr}{d\tau} \right)^2 + \frac{1}{2} \frac{l^2}{r^2} - GM \left( \frac{1}{r} + \frac{l^2}{r^3} \right) = \frac{1}{2} \left( e^2 - 1 \right) \tag{1}
\]

We can use this to derive an equation for \( \frac{dr}{dt} \), the rate of change of \( r \) with respect to the Schwarzschild time coordinate. The coordinate \( t \) isn’t the time as measured by any particular object (that time is the proper time \( \tau \) in the reference frame of the object) so we wouldn’t expect it to be the same as \( \frac{dr}{d\tau} \).

To get the equation, we can use

\[
\frac{dr}{d\tau} = \frac{dr}{dt} \frac{dt}{d\tau} = \frac{dr}{dt} \left( 1 - \frac{2GM}{r} \right)^{-1} \tag{2}
\]

where the last line uses the definition of \( e \). Plugging this into the top equation we get

\[
\frac{dr}{dt} = \frac{1}{e} \left( 1 - \frac{2GM}{r} \right) \left[ e^2 - 1 + 2GM \left( \frac{1}{r} + \frac{l^2}{r^3} \right) - \frac{l^2}{r^2} \right]^{1/2} \tag{3}
\]

As \( r \) approaches \( 2GM \), \( \frac{dr}{dt} \rightarrow 0 \).

In the special case where we drop an object from rest at \( r = r_0 \), we can work out both \( \frac{dr}{dt} \) and \( \frac{dr}{d\tau} \). In this case, motion is radially inward so \( l = 0 \). To find \( e \), we use the fact that for an object at rest at \( r = r_0 \):
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\[ e = \left(1 - \frac{2GM}{r_0}\right) \frac{dt}{d\tau} \]  
\[ = \left(1 - \frac{2GM}{r_0}\right) u^t \]  
\[ = \left(1 - \frac{2GM}{r_0}\right) \left(1 - \frac{2GM}{r_0}\right)^{-1/2} \]  
\[ = \left(1 - \frac{2GM}{r_0}\right)^{1/2} \]

We have therefore

\[ \frac{dr}{dt} = \frac{1}{e} \left(1 - \frac{2GM}{r}\right) \left(e^2 - 1 + \frac{2GM}{r}\right)^{1/2} \]  
\[ = \frac{1 - \frac{2GM}{r}}{\sqrt{1 - \frac{2GM}{r_0}}} \sqrt{2GM \left(\frac{1}{r} - \frac{1}{r_0}\right)} \]  
\[ = \left(1 - \frac{2GM}{r}\right) \sqrt{\frac{2GM}{r}} \sqrt{\frac{r_0 - r}{r_0 - 2GM}} \]

From \[ \] we get

\[ \frac{dr}{d\tau} = \sqrt{e^2 - 1 + \frac{2GM}{r}} \]  
\[ = \sqrt{2GM \left(\frac{1}{r} - \frac{1}{r_0}\right)} \]

Comparing the two, we see that

\[ \frac{dr}{dt} = \frac{1 - \frac{2GM}{r}}{\sqrt{1 - \frac{2GM}{r_0}}} \frac{dr}{d\tau} \]

For the case where the object is released from rest at \( r_0 = \infty \), the speed at \( r = 6GM \) is

\[ \frac{dr}{d\tau} = \frac{1}{\sqrt{3}} \]

which agrees with the earlier calculation done by a different method.