EMBEDDING A 2-D CURVED SURFACE IN 3-D: THE COSINE

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A third example embedding a curved-space 2-d metric in 3-d flat space. This time, our 2-d metric is

\[ ds^2 = \frac{dr^2}{\cos^2(r/R)} + r^2 d\phi^2 \] (1)

Here, the \( \phi \) component is already in the required form, so we can equate this to the cylindrical metric directly:

\[ dz^2 + r^2 d\phi^2 + dr^2 = \frac{dr^2}{\cos^2(r/R)} + r^2 d\phi^2 \] (2)

\[ \frac{dz}{dr} = \pm \sqrt{\frac{1 - \cos^2(r/R)}{\cos^2(r/R)}} \] (3)

\[ = \pm \tan \left( \frac{r}{R} \right) \] (4)

This integrates directly to give

\[ z = \pm R \ln \left| \cos \left( \frac{r}{R} \right) \right| \] (5)

This integral is a bit problematic, as the logarithm is defined only for positive arguments, which is why we’ve put the absolute value in the answer. If the limits include a region where the cosine is zero, the log goes to infinity, so in our case here, \( 0 \leq r/R < \pi/2 \). (This also follows from the original metric, since the cosine is in the denominator.) Since the cosine is always \( \leq 1 \), the log is always negative.

The lobe of this surface obtained from taking the + sign above looks like this (there is an upper lobe which is a mirror image of the lower one):
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