EMBEDDING A 2-D CURVED SURFACE INTO 3-D: INVERSE COSH

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A final example embedding a curved-space 2-d metric in 3-d flat space. This time, our 2-d metric is

\[ ds^2 = d\rho^2 + (\rho^2 + b^2)\,d\phi^2 \]  

(1)

where \( b \) is a constant.

To convert the \( \phi \) component to the required form of \( r^2 d\phi^2 \), we define \( r = \sqrt{\rho^2 + b^2} \). Then

\[ dr = \frac{\rho}{\sqrt{\rho^2 + b^2}}\,d\rho \]  

(2)

\[ = \frac{\sqrt{r^2 - b^2}}{r}\,d\rho \]  

(3)

\[ d\rho = \frac{r}{\sqrt{r^2 - b^2}}\,dr \]  

(4)

The metric becomes:

\[ ds^2 = \frac{r^2\,dr^2}{r^2 - b^2} + r^2\,d\phi^2 \]  

(5)

We now equate this to the cylindrical metric:

\[ dz^2 + r^2\,d\phi^2 + dr^2 = \frac{r^2\,dr^2}{r^2 - b^2} + r^2\,d\phi^2 \]  

(6)

\[ \frac{dz}{dr} = \frac{b}{\sqrt{r^2 - b^2}} \]  

(7)

This integral can be done with software or looked up, and is
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\[ z = b \ln \left( r + \sqrt{r^2 - b^2} \right) \] (8)

\[ = b \ln \left[ b \left( \frac{r}{b} + \sqrt{\frac{r^2}{b^2} - 1} \right) \right] \] (9)

\[ = b \ln \left( \frac{r}{b} + \sqrt{\frac{r^2}{b^2} - 1} \right) + b \ln b \] (10)

From tables of inverse hyperbolic functions, we see that this is equivalent to

\[ z = \text{barcosh} \left( \frac{r}{b} \right) + b \ln b \] (11)

We can ignore the last term as it is just a constant and serves only to raise or lower the surface as a whole.

The surface is similar to Flamm’s paraboloid, although since it is derived from hyperbolic functions and not parabolas, it’s not a paraboloid.